1. 6.3.12.

Solution: The length of the 0.95-confidence interval is \(2(1.96)\sigma_0/\sqrt{n} = 5.544/\sqrt{n}\). If we need \(5.544/\sqrt{n} \leq 1\), then \(n \geq (5.544/1)^2 = 30.7\). Thus, we choose \(n = 31\).

2. 6.3.13.

Solution:

(a) 
\[
\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} x_i - n\bar{x}^2 = n\bar{x} - n\bar{x}^2 = n\bar{x}(1 - \bar{x}).
\]

(b) (There are two methods, but students just need to provide one).

The plug-in estimate of \(\sigma^2\) is \(\hat{\sigma}^2 = \bar{x}(1 - \bar{x})\). The estimator of \(\sigma^2\) based on \(s^2\) is
\[
\tilde{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{n\hat{\theta}(1 - \hat{\theta})}{n-1} = \frac{n\sigma^2}{n-1}.
\]

They are different.

In addition, we have the second method. Note that \(\bar{X} = T/n\), \(E(T) = n\theta\), and \(V(T) = n\theta(1 - \theta)\). We obtain \(E(\bar{X}) = \theta\) and \(V(\bar{X}) = \theta(1 - \theta)/n\). Thus, \(E(\bar{X}^2) = E^2(\bar{X}) + V(\bar{X}) = \theta^2 + \theta(1 - \theta)/n\). Then,
\[
E[\bar{X}(1 - \bar{X})] = E(\bar{X}) - E(\bar{X}^2) = \theta - (\theta^2 + \frac{\theta(1 - \theta)}{n}) = \theta(1 - \theta) - \frac{\theta(1 - \theta)}{n}.
\]

Therefore, they are different.

(c) 
\[
E(\hat{\sigma}^2) = \frac{n-1}{n} E(\tilde{\sigma}^2) = \frac{n-1}{n} \sigma^2 = \frac{(n-1)\theta(1 - \theta)}{n}.
\]

Thus,
\[
\text{Bias}(\hat{\sigma}^2) = \frac{(n-1)\theta(1 - \theta)}{n} - \theta(1 - \theta) = \frac{\theta(1 - \theta)}{n}.
\]

3. 6.3.14.

Solution: Since the 95% confidence interval contains 2, we conclude \(H_0 : \psi(\theta) = 2\) at 0.05 significance level.

4. 6.3.15.

Solution:

(a) \(E X_1 = \theta\) and \(V X_1 = \theta(1 - \theta)\). Thus \(x_1\) is an unbiased estimator of \(\theta\).

(b) \(E X_1^2 = V X_1 + E^2 X_1 = \theta(1 - \theta) + \theta^2 = \theta\). Thus, it is not an unbiased estimator of \(\theta^2\). You can also write \(E X_1^2 = E X_1 = \theta\) to show your conclusion (since \(X_1^2 = X_1\)).

5. 6.3.24.

Solution:
(a) 
\[ E[\alpha T_1 + (1 - \alpha)T_2] = \alpha E(T_1) + (1 - \alpha)E(T_2) = \psi(\theta). \]
Thus, \( \alpha T_1 + (1 - \alpha)T_2 \) is an unbiased estimator for any \( \alpha \in [0, 1] \).

(b) If \( T_1 \) and \( T_2 \) are independent, then
\[ V[\alpha T_1 + (1 - \alpha)T_2] = \alpha^2 V(T_1) + (1 - \alpha)^2 V(T_2). \]

(c) Differentiating with respect to \( \alpha \), we obtain
\[ \frac{\partial}{\partial \alpha} V[\alpha T_1 + (1 - \alpha)T_2] = 2\alpha V(T_1) - 2(1 - \alpha)V(T_2). \]
Let
\[ 2\alpha V(T_1) - 2(1 - \alpha)V(T_2) = 0. \]
We obtain
\[ \alpha_{\text{best}} = \frac{V(T_2)}{V(T_1) + V(T_2)}. \]
If \( V(T_1) \) is much larger than \( V(T_2) \), then \( \alpha \) is small indicating that we should put more weight on \( T_2 \).

(d) If \( T_1 \) and \( T_2 \) are not independent, then
\[ V[\alpha T_1 + (1 - \alpha)T_2] = \alpha^2 V(T_1) + (1 - \alpha)^2 V(T_2) + 2\alpha(1 - \alpha)\text{Cov}(T_1, T_2). \]
Differentiating with respect to \( \alpha \), we obtain
\[ \frac{\partial}{\partial \alpha} V[\alpha T_1 + (1 - \alpha)T_2] = 2\alpha V(T_1) - 2(1 - \alpha)V(T_2) + 2 - 4\alpha\text{Cov}(T_1, T_2), \]
implying that
\[ \alpha_{\text{best}} = \frac{V(T_2) - \text{Cov}(T_1, T_2)}{V(T_1) + V(T_2) - 2\text{Cov}(T_1, T_2)}. \]

6. 6.3.25.

**Solution:**
\[ P(\mu \leq \bar{X} + k(\sigma_0/\sqrt{n})) = P(\bar{X} \geq \mu - k(\sigma_0/\sqrt{n})) \]
\[ = 1 - \Phi\left(\frac{\mu - (\mu - k\sigma_0/\sqrt{n})}{\sigma_0/\sqrt{n}}\right) \]
\[ = 1 - \Phi(k). \]
Thus, \( k = z_{1-\gamma} \) (classical notation) or \( k = z_\gamma \) (textbook notation, either is fine).

7. 6.3.26.

**Solution:** Since \( \bar{X} \sim N(\mu, \sigma_0^2/n) \),
\[ \max_{\mu \in H_0} P_\mu(\bar{X} \geq \bar{x}) = \max_{\mu \leq \mu_0}[1 - \Phi\left(\frac{\bar{x} - \mu}{\sigma_0/\sqrt{n}}\right)] = 1 - \Phi\left(\frac{\bar{x} - \mu_0}{\sigma_0/\sqrt{n}}\right), \]
where the last equation holds since \( 1 - \Phi[(\bar{x} - \mu)/(\sigma_0/\sqrt{n})] \) is increasing in \( \mu \).