1. 4.2.2.

Solution: For any $x \in (0, 1)$, there is

$$P(X_n \leq x) = P(Y^n \leq x) = P(Y \leq x^{1/n}) = x^{1/n}.$$  

For any $\epsilon > 0$, there is

$$P(|X_n| \leq \epsilon) = P(X_n \leq \epsilon) = \epsilon^{1/n} \to 1$$  

as $n \to \infty$. Thus, $X_n \xrightarrow{P} 0$ as $n \to \infty$.

2. 4.2.4.

Solution: Since $\bar{Y} \sim N(2, 5/n)$, there is

$$P(Y_1 + Y_2 + \cdots + Y_n \geq n) = P(\bar{Y} \geq 1)$$  

$$\geq P(1 \leq \bar{Y} \leq 3)$$  

$$= 1 - P(|\bar{Y} - 2| \geq 1)$$  

$$\geq 1 - V(\bar{Y})$$  

$$= 1 - \frac{5}{n}.$$  

To make $P(Y_1 + Y_2 + \cdots + Y_n \geq n) > 0.999$, it is enough to make $1 - 5/n > 0.999$, implying that we can choose any $n > 5000$.

3. 4.2.5.

Solution: Since $E(X_1) = V(X_i) = 8$, we have $E(\bar{X}) = 8$ and $V(X_n) = 8/n$. By the Chebyshev inequality,

$$P(X_1 + \cdots + X_n \geq 9n) = P(\bar{X} \geq 9)$$  

$$\geq P(\bar{X} \geq 9) + P(\bar{X} \leq 7)$$  

$$= P(|\bar{X} - 8| \geq 1)$$  

$$\leq V(\bar{X})$$  

$$= \frac{8}{n}.$$  

To make $P(X_1 + \cdots + X_n \geq 9n)$, we need $8/n \leq 0.001$, implying that we can choose any $n > 8000$.

4. Let $X_1, \cdots, X_n$ be independent random variables with $E(X_i) = \mu$ and $V(X_i) = \sigma^2_i$. Suppose there exist positive $a$ and $b$ such that $a \leq \sigma^2_i \leq b$ for all $i$. Use the Chebyshev inequality to show that $\bar{X} \xrightarrow{P} \mu$ as $n \to \infty$.

Solution: Note that $E(\bar{X}) = \mu$ and

$$V(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^{n} V(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2_i.$$  

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We have $a/n \leq V(\bar{X}) \leq b/n$. For any $\epsilon > 0$,

$$P(\mu - \epsilon \leq \bar{X} \leq \mu + \epsilon) = 1 - P(|\bar{X} - \mu| \geq \epsilon)$$

$$\geq 1 - \frac{V(\bar{X})}{\epsilon^2}$$

$$\geq 1 - \frac{b}{n\epsilon^2}$$

$$\to 1,$$

as $n \to \infty$. Thus, $\bar{X} \xrightarrow{p} \mu$ as $n \to \infty$.

5. Let $X_1, \ldots, X_n$ be iid random variable with common mean $\mu$ and variance $\sigma^2$. Let $Y_n = \sum_{i=1}^{n}(1 + 2^{-i})X_i/n$. Compute $E(Y_n)$ and $V(Y_n)$. Use the Chebyshev inequality to show that $Y_n \xrightarrow{p} \mu$.

Solution:

$E(Y_n) = \sum_{i=1}^{n}(1 + 1/n)^2E(X_i) = \mu + (1/2 - n/2)\mu$.

and

$V(Y_n) = \sum_{i=1}^{n}(1 + 1/n)^2V(X_i) = 4\sigma^2/n$.

For any $\epsilon > 0$, there is

$$P\left(\left|Y_n - \frac{(1 - 2^{-n})\mu}{n}\right| \geq \epsilon\right) \leq \frac{V(Y_n)}{\epsilon^2} \leq \frac{4\sigma^2}{n\epsilon^2} \to 0,$$

implying that $Y_n \xrightarrow{p} (1 - 2^{-n})/n \mu$. Since $(1 - 2^{-n})\mu/n \to 0$, we conclude that $Y_n \xrightarrow{p} \mu$.

6. Use the Chebyshev inequality to show that if $Y_n \sim \chi^2_n$ then $Y_n/n \xrightarrow{p} 1$ as $n \to \infty$.

Solution: Note that $E(Y_n/n) = E(Y_n)/n = 1$ and $V(Y_n/n) = V(Y_n)/n^2 = 2/n$. For any $\epsilon > 0$, there is

$$P\left(\frac{1}{n} - 1 \geq \epsilon\right) \leq \frac{V(Y_n/n)}{\epsilon^2} = 2\epsilon n \to 0,$$

implying that $Y_n/n \xrightarrow{p} 1$ as $n \to \infty$.

7. Suppose $X_1, \ldots, X_n \sim Unif[0, \theta]$. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$.

(a) Find the CDF and PDF of $X_{(n)}$.

Solution: For any $x \in (0, \theta)$, there is

$$F(x) = P(X_{(n)} \leq x) = P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) = \prod_{i=1}^{n} P(X_i \leq x) = x^n/\theta^n.$$

Thus,

$$f(x) = F'(x) = \frac{nx^{n-1}}{\theta^n}.$$

(b) Compute $E(X_{(n)})$ and $V(X_{(n)})$.

Solution:

$$E(X_{(n)}) = \int_{0}^{\theta} x f(x) dx = \frac{n}{\theta^n} \int_{0}^{\theta} x^n dx = \frac{n\theta}{n+1}.$$
and
\[
V(X(n)) = E(X_n^2) - E^2(X(n)) \\
= \frac{n}{\theta^n} \int_0^\theta x^{n+1} dx - \frac{n^2 \theta^2}{(n+1)^2} \\
= \frac{n\theta^2}{n+2} - \frac{n^2 \theta^2}{(n+1)^2} \\
= \frac{1}{(n+1)^2(n+2)}.
\]

(c) Show that \( X(n) \xrightarrow{P} \theta \) as \( n \to \infty \).

\textit{Solution:} Let \( a_n = \theta - E(X(n)) = \theta/n + 1 \). Then \( \lim_{n \to \infty} a_n = 0 \). Since \( \lim_{n \to \infty} V(X_n) = 0 \),
by the Chebyshev inequality, we conclude \( X(n) - a_n \xrightarrow{P} \theta \). Then, \( X(n) \xrightarrow{P} \theta \).

(d) Let \( Y_n = \sqrt{n}[X(n) - E(X(n))] \). Use the Chebyshev inequality to show \( Y_n \xrightarrow{P} 0 \).

\textit{Solution:} Note that \( E(Y_n) = 0 \) and
\[
V(Y_n) = V\{\sqrt{n}[X(n) - E(X(n))]\} = nV(X(n)) = \frac{n^2 \theta^2}{(n+1)^2(n+2)} \to 0
\]
as \( n \to \infty \). For any \( \epsilon > 0 \), there is
\[
P(-\epsilon \leq Y_n \leq \epsilon) = 1 - P(|Y_n| \geq \epsilon) \geq 1 - \frac{V(Y_n)}{\epsilon^2} = 1 - \frac{n^2 \theta^2}{(n+1)^2(n+2)} \to 1.
\]
Thus, \( Y_n \xrightarrow{P} 0 \) as \( n \to \infty \).