1. 4.2.2.

Solution: For any $x \in (0,1)$, there is

$$P(X_n \leq x) = P(Y^n \leq x) = P(Y \leq x^{1/n}) = x^{1/n}.$$

For any $\epsilon > 0$, there is

$$P(|X_n| \leq \epsilon) = P(X_n \leq \epsilon) = \epsilon^{1/n} \rightarrow 1$$

as $n \rightarrow \infty$. Thus, $X_n \xrightarrow{p} 0$ as $n \rightarrow \infty$.

2. 4.2.4.

Solution: Since $Y \sim N(2,5/n)$, there is

$$P(Y_1 + Y_2 + \cdots + Y_n \geq n) = P(\bar{X} \geq 1)$$

$$\geq P(1 \leq \bar{X} \leq 3)$$

$$= 1 - P(|\bar{X} - 2| \geq 1)$$

$$\geq 1 - V(\bar{Y})$$

$$= 1 - \frac{5}{n}.$$

To make $P(Y_1 + Y_2 + \cdots + Y_n \geq n) > 0.999$, it is enough to make $1 - 5/n > 0.999$, implying that we can choose any $n > 5000$.

3. 4.2.5.

Solution: Since $E(X_1) = V(X_i) = 8$, we have $E(\bar{X}) = 8$ and $V(X_n) = 8/n$. By the Chebyshev inequality,

$$P(X_1 + \cdots + X_n \geq 9n) = P(\bar{X} \geq 9)$$

$$\geq P(\bar{X} \geq 9) + P(\bar{X} \leq 7)$$

$$= P(|\bar{X} - 8| \geq 1)$$

$$\leq V(\bar{X})$$

$$= \frac{8}{n}.$$

To make $P(X_1 + \cdots + X_n \geq 9n)$, we need $8/n \leq 0.001$, implying that we can choose any $n > 8000$.

4. Let $X_1, \cdots, X_n$ be independent random variables with $E(X_i) = \mu$ and $V(X_i) = \sigma_i^2$. Suppose there exist positive $a$ and $b$ such that $a \leq \sigma_i^2 \leq b$ for all $i$. Use the Chebyshev inequality to show that $\bar{X} \xrightarrow{p} \mu$ as $n \rightarrow \infty$.

Solution: Note that $E(\bar{X}) = \mu$ and

$$V(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^{n} V(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2.$$
We have \(a/n \leq V(\bar{X}) \leq b/n\). For any \(\epsilon > 0\),

\[
P(\mu - \epsilon \leq \bar{X} \leq \mu + \epsilon) = 1 - P(|\bar{X} - \mu| \geq \epsilon) \geq 1 - \frac{V(\bar{X})}{\epsilon^2} \geq 1 - \frac{b}{n\epsilon^2} \to 1,
\]
as \(n \to \infty\). Thus, \(\bar{X} \xrightarrow{P} \mu\) as \(n \to \infty\).

5. Let \(X_1, \ldots, X_n\) be iid random variable with common mean \(\mu\) and variance \(\sigma^2\). Let \(Y_n = \sum_{i=1}^{n}(1 + 2^{-i})X_i/n\). Compute \(E(Y_n)\) and \(V(Y_n)\). Use the Chebyshev inequality to show that \(Y_n \xrightarrow{P} \mu\).

Solution:

\[
E(Y_n) = \frac{1}{n} \sum_{i=1}^{n}(1 + \frac{1}{2^i})E(X_i) = \mu + \frac{(1 - 2^{-n})\mu}{n}
\]
and

\[
V(Y_n) = \frac{1}{n^2} \sum_{i=1}^{n}(1 + \frac{1}{2^i})^2V(X_i) \leq \frac{4\sigma^2}{n}.
\]
For any \(\epsilon > 0\), there is

\[
P(\left|\frac{Y_n}{\epsilon} - \frac{(1 - 2^{-n})\mu}{\epsilon}\right| \geq \epsilon) \leq \frac{V(Y_n)}{\epsilon^2} \leq \frac{4\sigma^2}{n\epsilon^2} \to 0,
\]
implying that \(Y_n - (1 - 2^{-n})/n \xrightarrow{P} \mu\). Since \((1 - 2^{-n})\mu/n \to 0\), we conclude that \(Y_n \xrightarrow{P} \mu\).

6. Use the Chebyshev inequality to show that if \(Y_n \sim \chi^2_n\) then \(Y_n/n \xrightarrow{P} 1\) as \(n \to \infty\).

Solution: Note that \(E(Y_n/n) = E(Y_n)/n = 1\) and \(V(Y_n/n) = V(Y_n)/n^2 = 2/n\). For any \(\epsilon > 0\), there is

\[
P\left(\left|\frac{Y_n}{n} - 1\right| \geq \epsilon\right) \leq \frac{V(Y_n/n)}{\epsilon^2} = \frac{2}{n\epsilon^2} \to 0,
\]
implying that \(Y_n/n \xrightarrow{P} 1\) as \(n \to \infty\).

7. Suppose \(X_1, \ldots, X_n \sim Uniform[0, \theta]\). Let \(X_{(n)} = \max\{X_1, \ldots, X_n\}\).

(a) Find the CDF and PDF of \(X_{(n)}\).

Solution: For any \(x \in (0, \theta)\), there is

\[
F(x) = P(X_{(n)} \leq x) = P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) = \prod_{i=1}^{n} P(X_i \leq x) = x^n/\theta^n.
\]
Thus,

\[
f(x) = F'(x) = \frac{n x^{n-1}}{\theta^n}.
\]
(b) Compute \(E(X_{(n)})\) and \(V(X_{(n)})\).

Solution:

\[
E(X_{(n)}) = \int_{0}^{\theta} x f(x) dx = \frac{n}{\theta^n} \int_{0}^{\theta} x^n dx = \frac{n\theta}{(n + 1)}.
\]
and
\[ V(X(n)) = E(X(n)^2) - E^2(X(n)) \]
\[ = \frac{n}{\theta} \int_0^\theta x^{n+1} dx - \frac{n^2 \theta^2}{(n+1)^2} \]
\[ = \frac{n \theta^2}{n+2} - \frac{n^2 \theta^2}{(n+1)^2} \]
\[ = \frac{1}{(n+1)^2(n+2)}. \]

(c) Show that \( X(n) \xrightarrow{p} \theta \) as \( n \to \infty \).

*Solution:* Let \( a_n = \theta - E(X_n) = \theta/n + 1 \). Then \( \lim_{n \to \infty} a_n = 0 \). Since \( \lim_{n \to \infty} V(X_n) = 0 \), by the Chebyshev inequality, we conclude \( X(n) - a_n \xrightarrow{p} \theta \). Then, \( X(n) \xrightarrow{p} \theta \).

(d) Let \( Y_n = \sqrt{n}[X(n) - E(X_n)] \). Use the Chebyshev inequality to show \( Y_n \xrightarrow{p} 0 \).

*Solution:* Note that \( E(Y_n) = 0 \) and
\[ V(Y_n) = V\{\sqrt{n}[X(n) - E(X_n)]\} = nV(X(n)) = \frac{n^2 \theta^2}{(n+1)^2(n+2)} \to 0 \]
as \( n \to \infty \). For any \( \epsilon > 0 \), there is
\[ P(-\epsilon \leq Y_n \leq \epsilon) = 1 - P(|Y_n| \geq \epsilon) \geq 1 - \frac{V(Y_n)}{\epsilon^2} = 1 - \frac{n^2 \theta^2}{(n+1)^2(n+2)} \to 1. \]

Thus, \( Y_n \xrightarrow{p} 0 \) as \( n \to \infty \).