1. 6.1.3.

Solution: The likelihood function is

\[
L(\theta) = \prod_{i=1}^{n} \theta e^{-\theta X_i} = \theta^n e^{-\theta \sum_{i=1}^{n} X_i} = \theta^n e^{-(n\theta)\bar{X}}.
\]

Therefore, \(\bar{x}\) is a sufficient statistic.

2. 6.1.6.

Solution: The likelihood function is

\[
L(\theta) = \prod_{i=1}^{n} \theta^{X_i}(1-\theta)^{1-X_i} = \theta^{\sum_{i=1}^{n} X_i}(1-\theta)^{n-\sum_{i=1}^{n} X_i}.
\]

Therefore, \(\sum_{i=1}^{n} X_i\) is a sufficient statistic. Since its dimension is equal to one, it is a minimal sufficient statistic (\(\bar{X}\) is another).

3. 6.1.7.

Solution: The likelihood function is

\[
L(\theta) = \prod_{i=1}^{n} \frac{\theta X_i}{X_i!} e^{-\theta} = \frac{e^{-n\theta}}{\prod_{i=1}^{n} X_i!} \theta^{\sum_{i=1}^{n} X_i}.
\]

Therefore, \(\sum_{i=1}^{n} X_i\) is a sufficient statistic. Since its dimension is one, it is also a minimal sufficient statistic (\(\bar{X}\) is another).

4. 6.1.11.

Solution: Generally we do not have

\[
\int_{0}^{1} L(\theta|x) d\theta = 1
\]
as \(f_\theta(x)\) is the PDF of \(x\) but not \(\theta\). In addition (not required), we can use an example to confirm it. Let \(X \sim \text{Exp}(\theta)\). Then, the likelihood function is

\[
L(\theta|x_0) = \theta e^{-\theta x_0}.
\]

Then,

\[
\int_{0}^{\infty} \theta e^{-\theta x_0} d\theta = \frac{1}{x_0^2} \int_{0}^{\infty} \theta x_0 e^{-\theta x_0} d\theta x_0
\]

\[
= \frac{1}{x_0^2} \int_{0}^{\infty} t e^{-t} dt
\]

\[
= \frac{1}{x_0^2},
\]

which is not a constant. Note that the assumption of the question assumes \(\theta[0, 1]\). We can use a transformation from \((0, \infty)\) to \((0, 1)\), e.g., \(\beta = \log(\theta/(1 + \theta))\). Therefore, the domain of \(\theta\) is not an issue in the example.
5. **6.1.12.**

**Solution:** The PMF is \( p_{\theta}(x) = \theta^x(1 - \theta) \). The likelihood function is

\[
L(\theta) = \prod_{i=1}^{n} \theta^{X_i}(1 - \theta) = (1 - \theta)^n \theta \sum_{i=1}^{n} X_i.
\]

Therefore, \( \sum_{i=1}^{n} X_i \) is a sufficient statistic and also an minimal sufficient statistic (\( \bar{X} \) is another).

6. **6.1.14.**

**Solution:** If the first needs \( L(\theta^2) \), then the second needs and the second needs \( L[(100 \theta^2)/100] \). Therefore, they are equivalent methods.

7. **6.1.22.**

**Solution:** (I provide the solution step-by-step, but it is not required for students. They just need to show that the first-order derivative is zero and the second-order derivative is negative).

Let \( \theta = (\mu, \sigma^2) \), \( X = \sum_{i=1}^{n} X_i/n \), \( \hat{\sigma}^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n \). The likelihood function is

\[
L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2} = \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} \left( \frac{1}{\sigma^2} \right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2}
\]

Then, the loglikelihood function is

\[
\ell(\theta) = \log L(\theta) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{n}{2\sigma^2} \left\{ \hat{\sigma}^2 + (\bar{X} - \mu)^2 \right\}.
\]

The first-order partial derivatives are

\[
\frac{\partial \ell(\theta)}{\partial \mu} = -\frac{n}{\sigma^2} (\bar{X} - \mu)
\]

and

\[
\frac{\partial \ell(\theta)}{\partial (\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{n\hat{\sigma}^2}{2(\sigma^2)^2}.
\]

Then, we obtain

\[
\ell(\theta) = \log L(\theta)|_{\mu=\bar{X}} = 0
\]

and

\[
\frac{\partial \ell(\theta)}{\partial (\sigma^2)}|_{\sigma^2=\hat{\sigma}^2} = 0.
\]
The root of the first-order partial derivatives is unique. To confirm it is maximum, we need to compute the second-order partial derivatives. We obtain

\[
\frac{\partial^2 \ell(\theta)}{\partial \mu^2} = -\frac{n}{\sigma^2}
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \mu^2 \partial (\sigma^2)} = -\frac{n}{\sigma^4}(\bar{X} - \mu)
\]

and

\[
\frac{\partial^2 \ell(\theta)}{\partial (\sigma^2)^2} = + \frac{n}{2(\sigma^2)^2} - \frac{n\hat{\sigma}^2}{(\sigma^2)^3}.
\]

We obtain

\[
\frac{\partial^2 \ell(\theta)}{\partial \mu^2} \bigg|_{\mu = \bar{X}, \sigma^2 = \hat{\sigma}^2} = -\frac{n}{\hat{\sigma}^2} < 0
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \mu^2 \partial (\sigma^2)} \bigg|_{\mu = \bar{X}, \sigma^2 = \hat{\sigma}^2} = 0,
\]

and

\[
\frac{\partial^2 \ell(\theta)}{\partial (\sigma^2)^2} \bigg|_{\mu = \bar{X}, \sigma^2 = \hat{\sigma}^2} = -\frac{n}{2(\hat{\sigma}^2)^2} < 0.
\]

Therefore, \((\bar{X}, \hat{\sigma}^2)\) is a maximizer.