1. 6.5.1.

**Solution:** Let $\theta = \sigma^2$. The PDF is

$$f_\theta(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu_0)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi} \theta^{1/2}} e^{-\frac{(x-\mu_0)^2}{2\theta}}.$$

Its logarithm is

$$\log f_\theta(x) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log \theta - \frac{(x-\mu_0)^2}{2\theta}.$$

The second-order partial derivative is

$$\frac{\partial^2 \log f_\theta(x)}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{(x-\mu_0)^2}{\theta^3}.$$

The Fisher information is

$$I(\theta) = -\mathbb{E} \left[ \frac{1}{2\theta^2} - \frac{(X-\mu_0)^2}{\theta^3} \right] = \frac{1}{2\theta^2} = \frac{1}{2\sigma^4}.$$

2. 6.5.2.

**Solution:** The PDF is

$$f_\theta(x) = \frac{\theta_0 x^{\alpha_0-1}}{\Gamma(\alpha_0)} e^{-\theta_0 x}.$$

Its logarithm is

$$\log f_\theta(x) = -\log \Gamma(\alpha_0) + \alpha_0 \log \theta + (\alpha_0 - 1) \log x - \theta x.$$

The second-order partial derivative is

$$\frac{\partial^2 \log f_\theta(x)}{\partial \theta^2} = \frac{-\alpha_0}{\theta^2}.$$

Thus, the Fisher information is

$$I(\theta) = -\mathbb{E} \left[ \frac{-\alpha_0}{\theta^2} \right] = \frac{\alpha_0}{\theta^2}.$$

3. 6.5.3.

**Solution:** The logarithm of the PDF is

$$\log f_\alpha(x) = \log[\alpha(1+x)^{-(\alpha+1)}] = \log \alpha - (\alpha + 1) \log(1 + x)$$

for $0 < x < \infty$. Its second-order partial derivative is

$$\frac{\partial^2 \log f_\alpha(x)}{\partial \alpha^2} = -\frac{1}{\alpha^2}.$$

Thus, the Fisher information is

$$I(\alpha) = \frac{1}{\alpha^2}.$$
4. 6.5.4.

**Solution:** The logarithm of the PMF is

\[ \log f_\lambda(x) = -\log x! + x \log \lambda - \lambda. \]

Its second-order partial derivative is

\[ \frac{\partial^2 \log f_\lambda(x)}{\partial \lambda^2} = -\frac{x}{\lambda^2}. \]

The Fisher information is

\[ \mathbb{E} \left[ \frac{\partial^2 \log f_\lambda(x)}{\partial \lambda^2} \right] = \frac{\mathbb{E}(X)}{\lambda^2} = \frac{1}{\lambda}. \]

Using \( \hat{\theta} = \bar{X} \) for the MLE, we have

\[ \sqrt{n}(\bar{X} - \lambda) \sim_{\text{approx}} N(0, \lambda). \]

Based on the data, we have \( n = 20 \) and \( \bar{X} = 9.65 \). Thus, the 95% confidence interval for \( \lambda \) is

\[ 9.65 \pm 1.96 \sqrt{\frac{9.65}{20}} = [8.2885, 11.0115]. \]

To test \( H_0 : \lambda = 11 \) against \( H_1 : \lambda \neq 11 \), we conclude \( H_0 : \lambda = 11 \) since the 95% confidence interval contains 11. In addition, we can also compute the test statistic

\[ T = \left| \frac{\bar{X} - 11}{\sqrt{11/n}} \right| = \left| \frac{9.65 - 11}{\sqrt{11/20}} \right| = 1.82 < 1.96. \]

Thus, we conclude \( H_0 : \lambda = 11 \). For power, we write the rejection region

\[ C = \{ \bar{X} < 11 - 1.96\sqrt{11/n} \text{ or } \bar{X} > 11 + 1.92\sqrt{11/n} \} \]

\[ = \{ \bar{X} \leq 9.546 \text{ or } \bar{X} > 12.454 \}. \]

If \( \lambda = 10 \), then \( \sqrt{n}(\bar{X} - 10) \sim_{\text{approx}} N(0, 10) \) and the power is

\[ P(\bar{X} \leq C|\lambda = 10) = P(\bar{X} \leq 9.546) + P(\bar{X} > 12.454) \]

\[ \approx \Phi\left( \frac{9.546 - 10}{\sqrt{10/20}} \right) + \left[ 1 - \Phi\left( \frac{12.454 - 10}{\sqrt{10/20}} \right) \right] \]

\[ = \Phi(-0.64) + [1 - \Phi(3.47)] \]

\[ = 0.2613. \]

5. 6.5.5.

**Solution:** The logarithm of the PDF is

\[ \log f_\theta(x) = \log(\theta^2xe^{-\theta x}) = 2 \log \theta + \log x - \theta x. \]

Its second-order partial derivative is

\[ \frac{\partial^2 \log f_\theta(x)}{\partial \theta^2} = -\frac{2}{\theta^2} \Rightarrow I(\theta) = \frac{2}{\theta^2}. \]

Using the MLE of \( \theta \) as \( \hat{\theta} = 2/\bar{X} \), we obtain

\[ \sqrt{n}(\hat{\theta} - \theta) = \sqrt{n}(\bar{X}/2 - \theta) \sim_{\text{approx}} N(0, \theta^2/2). \]

2
Based on the data, we obtain $\bar{X} = 1627.47$ and $\hat{\theta} = 0.001229$. Thus, the 90% confidence interval is

$$\hat{\theta} \pm 1.645 \sqrt{\frac{\hat{\theta}^2}{2n}} = 0.001229 \pm 1.645 \sqrt{\frac{0.001229^2}{54}} = [0.0009539, 0.001504].$$

6. 6.5.7.

**Solution:** The PDF is

$$f(x) = \alpha(1 + x)^{-\alpha - 1}.$$ 

Then,

$$E[X] = \int_0^\infty \alpha x(1 + x)^{-\alpha - 1} dx$$

$$= \int_0^\infty \alpha[(1 + x) - 1](1 + x)^{-\alpha - 1} dx$$

$$= \int_0^\infty \alpha(1 + x)^{-\alpha} dx - \int_0^\infty \alpha(1 + x)^{-\alpha - 1} dx$$

$$= \frac{\alpha}{1 - \alpha} (1 + x)^{1 - \alpha}|_0^\infty - \alpha$$

$$= \frac{-\alpha}{1 - \alpha} - 1$$

$$= \frac{1}{\alpha - 1}.$$ 

The loglikelihood function is

$$\ell(\alpha) = n \log \alpha - (\alpha + 1) \sum_{i=1}^n \log(1 + X_i) \Rightarrow \ell'(\alpha) = \frac{n}{\alpha} - \sum_{i=1}^n \log(1 + X_i).$$

Thus, the MLE is

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^n \log(1 + X_i)}.$$ 

Based on the data, we choose $X_i$ as the values of income minus 20. We obtain $n = 20$ and $\hat{\alpha} = 1.7860$. Thus, the estimate of the Fisher information is $I(\hat{\alpha}) = 1/\hat{\alpha}^2$. The 95% confidence interval for $\alpha$ is

$$\hat{\alpha} \pm 1.96 \sqrt{\frac{1}{I(\hat{\alpha})n}} = \hat{\alpha} \pm 1.96 \sqrt{\frac{\hat{\alpha}^2}{20}}$$

$$= 1.7860 \pm 1.96 \times \frac{1.7860}{\sqrt{20}}$$

$$= [1.0033, 2.5687].$$ 

Using $\mu = E(X) = 1/(\alpha - 1)$, we obtain $\alpha = 1.2$ if $X + 25 - 20 = 5$. Since 1.2 is inside the confidence interval, we accept $H_0 : \mu = 5$ (i.e., conclude that mean income is 25).

**Remark:** It is OK if the student uses the original data. It should provide the 95% confidence interval as $[0.1823, 0.4666]$. If mean is 25, then $\alpha = 1.04$, indicating that $H_0$ is reject.

7. 6.5.8.

**Solution:** The loglikelihood function is

$$\ell(\theta) = \sum_{i=1}^n \log \theta e^{-\theta X_i} = n \log \theta - n \bar{X} \theta.$$
We obtain
\[ \ell'(\theta) = \frac{n}{\theta} - n\bar{X} \Rightarrow \hat{\theta} = \frac{1}{\bar{X}}. \]
Then
\[ \log f_\theta(x) = \log \theta - x\theta \Rightarrow \frac{\partial^2 \log f_\theta(x)}{\partial \theta^2} = -\frac{1}{\theta^2} - x \Rightarrow I(\theta) = \frac{1}{\theta^2}. \]
Thus,
\[ \sqrt{n}(\hat{\theta} - \theta) \sim_{\text{approx}} N(0, \theta^2), \]
implying that an approximate left-sided \( \gamma \) confidence interval for \( \theta \) is
\[ (-\infty, \hat{\theta} + z_\gamma \hat{\theta}/\sqrt{n}]. \]