1. 7.1.1.

**Solution:** The conditional PMF is

\[ f_\theta(x) = \theta^I(x=1)(1 - \theta)^I(x=2). \]

The joint PMF is

\[ f_\theta(X) = \theta^I(X_1=1)+I(X_2=1)(1 - \theta)^I(X_1=2)+I(X_2=2)\pi(\theta_j), \]

where \( \pi(\theta_j) \) is given by \( \pi(1) = 0.2, \pi(2) = 0.4, \) and \( \pi(3) = 0.4. \) The marginal PMF is

\[ \tilde{f}(X) = \sum_{j=1}^{3} f_\theta(X)\pi(\theta_j), \]

The posterior PMF of \( \theta \) is

\[ q(\theta|X) = \frac{f_\theta(X)\pi(\theta)}{\tilde{f}(X)}. \]

We obtain the following Table.

<table>
<thead>
<tr>
<th>Joint for ( \theta )</th>
<th>Posterior for ( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1/10</td>
</tr>
<tr>
<td>2</td>
<td>1/10</td>
</tr>
</tbody>
</table>

2. 7.1.2.

**Solution:** The posterior \( \theta \) follows \( \text{Beta}(n\bar{x} + \alpha, n(1 - \bar{x}) + \beta) \). Therefore,

\[ E(\theta|x) = \frac{n\bar{x} + \alpha}{n + \alpha + \beta} \]

and

\[ V(\theta|x) = \frac{(n\bar{x} + \alpha)[n(1 - \bar{x}) + \beta]}{(n + \alpha + \beta)^2(n + \alpha + \beta + 1)}. \]

3. 7.1.4.

**Solution:** The likelihood function given \( \lambda \) is

\[ f_\lambda(X_1, \ldots, X_n) = \prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!}e^{-\lambda} = \frac{\lambda^{n\bar{X}}}{\prod_{i=1}^{n} X_i!}e^{-n\lambda}. \]

The prior PDF for \( \theta \) is

\[ \pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1}e^{-\beta\lambda}. \]

The joint PMF-PDF of \( (X_1, \ldots, X_n, \lambda) \) is

\[ f_\lambda(X_1, \ldots, X_n)\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)\prod_{i=1}^{n} X_i!}\lambda^{n\bar{X}+\alpha-1}e^{-(n+\beta)\lambda}. \]

The marginal PMF of \( (X_1, \ldots, X_n) \) is

\[ \tilde{f}(X_1, \ldots, X_n) = \int_{0}^{\infty} f_\lambda(X_1, \ldots, X_n)\pi(\lambda)d\lambda = \frac{\beta^\alpha}{\Gamma(\alpha)\prod_{i=1}^{n} X_i!} \frac{\Gamma(n\bar{X} + \alpha)}{(n + \beta)^{n\bar{X}+\alpha}}. \]
The posterior PDF of $\theta$ is

$$q(\lambda|X_1, \cdots, X_n) = \frac{(n + \beta)^{nX + \alpha}}{\Gamma(nX + \alpha)} \lambda^{nX + \alpha - 1} e^{-(n + \beta)\lambda},$$

which is the PDF of $\Gamma(nX + \alpha, n + \beta)$.

4. 7.1.5.

**Solution:** The PDF of $x = (x_1, \cdots, x_n)$ is

$$f_\theta(x) = \frac{1}{\theta^n} \prod_{i=1}^n I(0 \leq x_i \leq X_n) = \frac{1}{\theta^n} I(0 \leq x_{(1)} \leq x_{(n)} I(x_{(1)} \leq x_{(n)} \leq \theta),$$

where $x_{(1)} = \min_{i \leq n}(x_i)$ and $x_{(n)} = \max_{i \leq n}(x_i)$. The prior PDF of $\theta$ is

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta}.$$

The joint PDF of $(x, \theta)$ is

$$f_\theta(x)\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} I(0 \leq x_{(1)} \leq x_{(n)} I(x_{(1)} \leq x_{(n)} \leq \theta) \theta^{\alpha-1} n e^{-\beta \theta}.$$

The marginal PDF of $x$ is

$$\tilde{f}(x) = \int_0^\infty f_\theta(x)\pi(\theta) d\theta = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{x_{(n)}}^\infty \theta^{\alpha-1} n e^{-\beta \theta} d\theta.$$

The posterior PDF of $\theta$ is

$$q(\theta|x) = \frac{\theta^{\alpha-1} n e^{-\beta \theta}}{\int_{x_{(n)}}^\infty \theta^{\alpha-1} n e^{-\beta \theta} d\theta}$$

for $\theta \geq x_{(n)}$.

5. 7.1.9.

**Solution:**

(a) Using $T = \sum_{i=1}^n X_i \sim Bin(n, \theta)$, the PMF of $T$ given $\theta$ is

$$f_\theta(T) = \binom{n}{T} \theta^T (1 - \theta)^{n-T}, 0.4 \leq \theta \leq 0.6.$$

The prior PDF for $\theta$

$$\pi(\theta) = 5, 0.4 \leq \theta \leq 0.6.$$

The joint PMF-PDF of $(T, \theta)$ is

$$f_\theta(T)\pi(\theta) = \frac{5(n!)}{T!(n-T)!} \theta^T (1 - \theta)^{n-T}.$$

The posterior PDF of $\theta$ is

$$q(\theta|T) = \frac{\theta^T (1 - \theta)^{n-T}}{\int_{0.4}^{0.6} \theta^T (1 - \theta)^{n-T} d\theta}, 0.4 \leq \theta \leq 0.6.$$
(b) Since the true value is outside $[0.4, 0.6]$, the posterior density of $\theta$ does not put any probability mass around 0.99.

(c) The prior density must be positive in the neighborhood of the true value of $\theta$.

6. 7.2.10.

Solution: The PDF of $\mathbf{x} = (X_1, \cdots, X_n)$ given $\theta$ is

$$f_\theta(\mathbf{x}) = \prod_{i=1}^{n} \theta e^{-\theta X_i} = \theta^n e^{-n \bar{X}\theta}.$$ 

The prior for $\theta$ is

$$\pi(\theta) = \frac{\beta^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{\alpha_0-1} e^{-\beta \theta}.$$ 

The joint PDF of $(\mathbf{x}, \theta)$ is

$$f_\theta(\mathbf{x})\pi(\theta) = \frac{\beta^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{n+\alpha_0-1} e^{-(n \bar{X} + \beta)\theta}.$$ 

The posterior PDF of $\theta$ given $\mathbf{x}$ is

$$q(\theta|\mathbf{x}) = \frac{f_\theta(\mathbf{x})\pi(\theta)}{f_\theta(\mathbf{x})\pi(\theta)d\theta} = \frac{(n\bar{X} + \beta)^{n+\alpha_0}}{\Gamma(n + \alpha_0)} \theta^{n+\alpha_0-1} e^{-(n \bar{X} + \beta)\theta},$$

which is the PDF of $\Gamma(n + \alpha_0, n\bar{X} + \beta)$. Then

$$E(\theta|\mathbf{x}) = \frac{n + \alpha_0}{n\bar{X} + \beta}$$

and

$$V(\theta|\mathbf{x}) = \frac{n + \alpha_0}{(n\bar{X} + \beta)^2}.$$ 

To compute the posterior, we consider

$$\log q(\theta|\mathbf{x}) = \log \frac{(n\bar{X} + \beta)^{n+\alpha_0}}{\Gamma(n + \alpha_0)} + (n + \alpha_0 - 1) \log \theta - (n \bar{X} + \beta)\theta.$$ 

Let its first-order derivative be zero. We have

$$\frac{n + \alpha_0 - 1}{\theta} - (n \bar{X} + \beta) = 0 \Rightarrow \theta_{\text{mode}} = \frac{n + \alpha_0 - 1}{n \bar{X} + \beta}.$$