There are totally 37 points in the exam. The students with score higher than or equal to 35 points will receive 35 points. Please write down your name, student ID number below.

NAME: ________________________________
ID: ________________________________
1. (17 points, 1 point each). Fill in the blanks.

(a) (4 points). Let \( X \) be a random variable with \( E(X) = 1.5 \) and \( E(X^2) = 5 \). Let \( Y = -3X + 2 \). Then, \( V(X) = \), \( E(Y) = \), \( V(Y) = \), and \( \text{Cov}(X,Y) = \).

(b) (3 points). Let \( X \) be a random variable with PDF \( f(x) = 2x \) if \( 0 \leq x \leq 1 \) or \( f(x) = 0 \) otherwise. Then, \( E(X) = \), \( V(X) = \), and \( E(X^4) = \).

(c) (3 points). Let \( (X, Y) \) be bivariate random vector with the joint PDF \( f(x,y) = 3y^2 \) for \( 0 \leq x, y \leq 1 \). Then, the marginal PDF of \( X \) is \( f_X(x) = \), the marginal PDF of \( Y \) is \( f_Y(y) = \), and \( \text{Cov}(X,Y) = \).

(d) (4 points). Let \( X_1, \ldots, X_{100} \) be iid random variables common PMF

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Let \( \mu = E(X_i) \) and \( \sigma^2 = V(X_i) \). Then, \( \mu = \) and \( \sigma^2 = \). By the central limit theorem, the approximate distribution of \( \bar{X} \) is \( N(\mu, \sigma^2/n) \) and \( P(\bar{X} \leq 2.55) \approx \) \( \text{normal} \).

(e) (3 points). Let \( X \sim \text{Bin}(1000, 0.45) \). Then, \( E(X) = \), \( V(X) = \). By the central limit theorem, we have \( P(X \leq 470) \approx \) \( \text{normal} \).

Answer: (a) 2.75, -2.5, 24.75, -8.25; (b) 2/3, 1/18, 1/3; (c) 1 if \( 0 < x < 1 \), \( 3y^2 \) if \( 0 < y < 1 \), 0; (d) 2.4, 1.24, \( N(2.4, 0.0124) \), 0.9110; (e) 450, 247.5, 0.8982.
2. (8 points). Let $\mathbf{x} = (X_1, X_2, X_3, X_4)^\top$ be a four dimensional normal random vector with

$$
\mu = \mathbb{E}(\mathbf{x}) = \begin{pmatrix}
1 \\
-1 \\
-1 \\
1
\end{pmatrix}, \quad \Sigma = \text{Cov}(\mathbf{x}) = \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 0 & 1 & 4
\end{pmatrix}.
$$

Let

$$
\mathbf{y} = \begin{pmatrix}
X_1 + 2X_2 - X_3 - X_4 \\
2X_1 - X_2 + X_3 + 2X_4
\end{pmatrix}.
$$

(a) (2 points). Provide the distribution of $\bar{X} = (X_1 + X_2 + X_3 + X_4)/4$.

**Solution:**

$$
\mathbb{E}(\bar{X}) = \frac{1}{4}[\mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4)] = \frac{1}{4}(1 - 1 - 1 + 1) = 0
$$

and

$$
\text{V}(\bar{X}) = \frac{1}{16}[\text{V}(X_1) + \text{V}(X_2) + 2\text{Cov}(X_1, X_2) + \text{V}(X_3) + \text{V}(X_4) + 2\text{Cov}(X_3, X_4)]
$$

$$
= \frac{1}{16}(2 + 2 + 2 + 4 + 4 + 2)
$$

$$
= 1.
$$

Thus, the distribution of $\bar{X}$ is $N(0, 1)$.

**Note:** You can also use the matrix method to compute $\mathbb{E}(\bar{X})$ and $\text{V}(\bar{X})$. This method is also recommended.

(b) (2 points). Provide the distribution of $\mathbf{y}$.

**Solution:** Let

$$
\mathbf{A} = \begin{pmatrix}
1 & 2 & -1 & -1 \\
2 & -1 & 1 & 2
\end{pmatrix}.
$$

Then, $\mathbf{y} = \mathbf{A}\mathbf{x}$. We have

$$
\mathbb{E}(Y_1) = \mathbf{A}\mu = \begin{pmatrix}
1 & 2 & -1 & -1 \\
2 & -1 & 1 & 2
\end{pmatrix}\begin{pmatrix}
1 \\
-1 \\
-1 \\
1
\end{pmatrix} = \begin{pmatrix}
-1 \\
-4
\end{pmatrix}
$$

and

$$
\text{V}(Y_1) = \mathbf{A}\Sigma\mathbf{A}^\top = \begin{pmatrix}
1 & 2 & -1 & -1 \\
2 & -1 & 1 & 2
\end{pmatrix}\begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 0 & 1 & 4
\end{pmatrix}\begin{pmatrix}
1 & 2 \\
2 & -1 \\
-1 & 1 \\
-1 & 2
\end{pmatrix} = \begin{pmatrix}
24 & -12 \\
-12 & 30
\end{pmatrix}.
$$

Thus,

$$
\mathbf{y} \sim N\left(\begin{pmatrix}
-1 \\
4
\end{pmatrix}; \begin{pmatrix}
24 & -12 \\
-12 & 30
\end{pmatrix}\right).
$$
(c) (2 points). Justify whether the first and second components of $y$ are independent. Show your work.

**Solution:** Since the covariance between them is not zero, we conclude that the first and second components of $y$ are not independent.

(d) (2 points). Find $a$ and $b$ such that $X_2$ and $X_1 + aX_2$ are independent, and $X_4$ and $X_3 + bX_4$ are independent.

**Solution:**

\[
\text{Cov}(X_2, X_1 + aX_2) = \text{Cov}(X_2, X_1) + aV(X_2) = 1 + 2a = 0 \Rightarrow a = -1/2,
\]

we conclude that $X_1$ and $X_1 + aX_2$ are independent if $a = -1/2$. By

\[
\text{Cov}(X_4, X_3 + bX_4) = \text{Cov}(X_4, X_3) + bV(X_4) = 1 + 4b = 0 \Rightarrow b = -1/4,
\]

we conclude that $X_4$ and $X_3 + bX_4$ are independent if $a = -1/4$.

3. (6 points). Use the Chebyshev inequality to show the following problems.

(a) (2 points). Let $X_1, \ldots, X_n$ be iid random variables with $E(X_i) = 0$, $V(X_i) = \sigma^2 > 0$. Let $Y = \sum_{i=1}^{n} iX_i/n^2$. Show that $Y \overset{P}{\to} 0$ as $n \to \infty$.

**Solution:** By $E(Y_i) = n^{-2} \sum_{i=1}^{n} iE(X_i) = 0$ and

\[
V(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} i^2 V(X_i) = \frac{\sigma^2}{n^4} \sum_{i=1}^{n} i^2 = \frac{\sigma^2 n(n+1)(2n+1)}{6n^4} \to 0
\]

as $n \to \infty$, for any $\epsilon > 0$, we have

\[
P(|Y| \geq \epsilon) = \frac{V(Y)}{\epsilon^2} \to 0
\]

as $n \to \infty$. Thus, $Y \overset{P}{\to} 0$ as $n \to \infty$.

(b) (2 points). Let $X_n \sim \chi^2_n$. Show that $X_n/n \overset{P}{\to} 1$ as $n \to \infty$.

**Solution:** By $E(X_n) = n$ and $V(X_n) = 2n$, we have $E(X_n/n) = 1$. For any $\epsilon > 0$, we have

\[
P(|X_n/n - 1| \geq \epsilon) \leq \frac{1}{\epsilon^2} V(X_n/n) = \frac{1}{n^2 \epsilon^2} V(X_n) = \frac{2}{n \epsilon} \to \infty
\]

as $n \to \infty$, implying that $X_n/n \overset{P}{\to} 1$ as $n \to \infty$.

(c) (2 points). Let $X_1, \ldots, X_n$ be iid with $\mu = E(X_i) = 0$ and $\sigma^2 = V(X_i) > 0$. Let $Y = n^\alpha \bar{X}$ with $0 < \alpha < 1/2$. Show that $Y \overset{P}{\to} 0$ as $n \to \infty$.

**Solution:** By $E(\bar{X}) = 0$ and $V(\bar{X}) = \sigma^2/n$, we have $E(Y) = 0$ and

\[
V(Y) = n^{2\alpha} V(\bar{X}) = \frac{n^{2\alpha} \sigma^2}{n} \to 0
\]

if $0 < \alpha < 1/2$. For any $\epsilon > 0$, there is

\[
P(|Y| \geq \epsilon) = \frac{V(Y)}{\epsilon^2} \to 0
\]

as $n \to \infty$. Thus, $Y \overset{P}{\to} 0$ as $n \to \infty$. 

4
4. (8 points). Provide the maximum likelihood estimator (MLE) of the following problems.

(a) (2 points). Let \( X_1, \ldots, X_n \) be iid \( N(0, \theta) \), \( \theta > 0 \). Find the MLE of \( \theta \).

Solution: The loglikelihood function of \( \theta \) is
\[
\ell(\theta) = \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{X_i^2}{2\theta}} = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \theta - \frac{1}{2\theta} \sum_{i=1}^{n} X_i^2.
\]
Then,
\[
\ell'(\theta) = -\frac{n}{2\theta} - \frac{1}{2\theta^2} \sum_{i=1}^{n} X_i^2 = 0 \Rightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2.
\]
Thus, the MLE of \( \theta \) is \( \hat{\theta} = \frac{\sum_{i=1}^{n} X_i^2}{n} \).

(b) (2 points). Let \( X_1, \ldots, X_n \) be iid \( Bin(10, \theta) \), \( 0 < \theta < 1 \). Find the MLE of \( \theta \).

Solution: The PMF of \( X_i \) is \( (10)^{X_i} \theta^{X_i} (1 - \theta)^{10 - X_i} \). The loglikelihood function of \( \theta \) is
\[
\ell(\theta) = \sum_{i=1}^{n} \left[ \log \left( \frac{10}{X_i} \right) + X_i \log \theta + (10 - X_i) \log(1 - \theta) \right].
\]
Then,
\[
\ell'(\theta) = \frac{1}{\theta} \sum_{i=1}^{n} X_i - \frac{1}{1 - \theta} \sum_{i=1}^{n} (10 - X_i) = \frac{n\bar{X}}{\theta} - \frac{10n - n\bar{X}}{1 - \theta} = 0 \Rightarrow \hat{\theta} = \frac{\bar{X}}{10}.
\]
Thus, the MLE of \( \theta \) is \( \hat{\theta} = \frac{\bar{X}}{10} \).

(c) (2 points). Let \( X_1, \ldots, X_n \) be iid with common PDF \( f(x) = 4x^3/\theta^4 \), \( 0 \leq x \leq \theta \). Find the MLE of \( \theta \).

Solution: The likelihood function of \( \theta \) is
\[
L(\theta) = \prod_{i=1}^{n} \frac{4X_i^2}{\theta^4} I(0 \leq X_i \leq \theta) = \frac{4\prod_{i=1}^{n} X_i^3}{\theta^4} I(\max_{1 \leq i \leq n} (X_i) \leq \theta) = \frac{4\prod_{i=1}^{n} X_i^3}{\theta^4} I(X_{(n)} \leq \theta).
\]
Thus, the MLE is \( \hat{\theta} = X_{(n)} \).

(d) (2 points). Let \( X_1, \ldots, X_n \) be iid \( N(\mu, \sigma^2) \). Find the MLE of \( E(X_i^2) \).

Solution: The loglikelihood function of \( \theta = (\mu, \sigma^2) \) is
\[
\ell(\theta) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2.
\]
Maximizing \( \ell(\theta) \), we obtain the MLE of \( \mu \) is \( \hat{\mu} = \bar{X} \) and the MLE of \( \sigma^2 \) is \( \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n} \).
Note that \( E(X_i) = V(X_i) + E^2(X_i) = \sigma^2 + \mu^2 \). The MLE of \( E(X_i) \) is \( \hat{E}^2(X_i) = \hat{\sigma}^2 + \bar{X}^2 = \frac{\sum_{i=1}^{n} X_i^2}{n} \).