There are totally 42 points in the exam. The students with score higher than or equal to 40 points will receive 40 points. Please write down your name and student ID number below.

NAME: ________________________________
ID: ________________________________
1. (20 points) (1 point each). Fill in the blanks. Answers must be put in the blanks.

(a) (4 points). Let \((X_1, X_2)\) be bivariate random variables with \(E(X_1) = 2\), \(E(X_2) = 3\), \(V(X_1) = 1.5\), \(V(X_2) = 2.4\), and \(\text{Cov}(X_1, X_2) = 1.2\). Let \(Y_1 = 2X_1 + X_2\) and \(Y_2 = X_1 - X_2\). Then, \(E(Y_1) = \ldots\), \(V(Y_1) = \ldots\), \(V(Y_2) = \ldots\), and \(\text{Cov}(Y_1,Y_2) = \ldots\).

(b) (3 points). Let \(X_1, \ldots, X_{100}\) be iid random variables with common mean \(\mu = 0.5\) and variance \(\sigma^2 = 2.89\). By the central limit theorem, the distribution of \(X\) is approximately \ldots, \(P(0.3 \leq X \leq 0.7) \approx \ldots\), and \(P(X \leq 0.8) \approx \ldots\).

(c) (3 points). Let \(X_1, \ldots, X_n\) be iid \(N(\mu, 4)\). From the data, there is \(\bar{X} = 3.6\). If \(n = 20\), then the 95% confidence interval for \(\mu\) is \ldots. If \(n = 50\), then the 95% confidence interval for \(\mu\) is \ldots. If we want to make the length of the 95% confidence interval not over 0.5, we need the sample size \(n\) at least \ldots.

(d) (3 points). Let \(X_1, \ldots, X_{15}\) be iid \(N(\mu, \sigma^2)\). From the data, we have \(\bar{x} = 4.25\) and \(s^2 = 3.12\). Then, the 95% confidence interval for \(\mu\) is \ldots, the 95% confidence interval for \(\sigma^2\) is \ldots, and the 95% confidence interval for \(\sigma\) is \ldots. (\(\chi^2_{0.975,14} = 5.629, \chi^2_{0.025,14} = 26.119,\) and \(\chi^2_{0.025,15} = 27.488\)).

(e) (4 points). Let \(X \sim \text{Bin}(n, p)\), where the observed value of \(X\) is \(x = 39\) when \(n = 100\). By the central limit theorem based on \(n = 100\), the approximate distribution of \(\hat{p} = X/n = X/100\) is \ldots. Based on the data, the maximum likelihood estimate of \(\hat{p}\) is \ldots. Based on the data, the 95% confidence interval for \(p\) is \ldots. To ensure the length of the 95% confidence interval not over 0.02, we need \(n\) at least \ldots.

(f) (3 points). Let \(X_1, \ldots, X_n \sim N(\mu, 1)\). Suppose we want to test

\[H_0 : \mu \leq 2 \iff H_1 : \mu > 2.\]

If \(n = 10\) and \(\bar{X} = 2.63\), then we \ldots \(H_0\) at 0.05 significance level and \ldots \(H_0\) at 0.01 significance level. The P-value of the test is \ldots.

Solution: (a) 7; 13.2; 1.5; -0.6. (b) \(N(0.5, 0.0289); 0.7606; 0.9612\). (c) [2.723, 4.477]; [3.046, 4.154]; 246. (d) [3.272, 5.228]; [1.672, 7.762]; [1.293, 2.786]. (e) \(N(p,p(1-p)/100); 0.39; 0.2944, 0.4856\); 9604. (f) reject; accept; 0.0232.
2. (6 points). Let $X_1, \ldots, X_{10}$ be iid $N(\mu, 1.6)$. Suppose we want to test

$$
H_0 : \mu \leq 3 \leftrightarrow H_1 : \mu > 3.
$$

Suppose the rejection region is $C = \{ \bar{X} \geq 3.9 \}$.

(a) (2 points). Compute the type I error probability when $\mu = 2.8$.

Solution: Note that $\bar{X} \sim N(\mu, 0.16)$. The type I error probability is

$$
P(\bar{X} \geq 3.9|\mu = 2.8) = P(N(\mu, 0.16) \geq 3.9|\mu = 2.8)
= P(N(2.8, 0.16) \geq 3.9)
= 1 - \Phi\left(\frac{3.9 - 2.8}{\sqrt{0.16}}\right)
= 1 - \Phi(2.75)
= 0.0030.
$$

(b) (2 points). Compute the type II error probability when $\mu = 4.0$.

Solution: The type II error probability is

$$
P(\bar{X} \leq 3.9|\mu = 4.0) = P(N(\mu, 0.16) \leq 3.9|\mu = 4.0)
= P(N(4.0, 0.16) \leq 3.9)
= \Phi\left(\frac{3.9 - 4.0}{\sqrt{0.16}}\right)
= \Phi(-0.25)
= 0.4013.
$$

(c) (2 points). Derive the significance level of the test.

Solution: The significance level is

$$
P(\bar{X} \geq 3.9|\mu = 3.0) = P(N(\mu, 0.16) \geq 3.9|\mu = 3.0)
= 1 - P(N(3.0, 0.16) \leq 3.9)
= 1 - \Phi\left(\frac{3.9 - 3.0}{\sqrt{0.16}}\right)
= \Phi(2.25)
= 0.0122.
$$
3. (4 points). Provide maximum likelihood estimation in the following problem.

(a) (2 points). Let $X_1, \cdots, X_n$ be iid with common PDF $f_\theta(x) = \frac{2^{3/2} x^{3/2}}{\sqrt{\pi}} e^{-x^2}$. Find the MLE of $\theta$.

**Solution:** The loglikelihood function is

$$\ell(\theta) = \sum_{i=1}^{n} \log \frac{2^{3/2} X_i^{3/2}}{\sqrt{\pi}} e^{-x_i^2} = \frac{n}{2} \log \frac{4}{\pi} + \frac{3}{2} \sum_{i=1}^{n} \log X_i + \frac{n}{2} \log \theta - n \theta \bar{X}.$$

Then,

$$\ell'(\theta) = \frac{n}{2\theta} - n \bar{X} \Rightarrow \hat{\theta} = \frac{1}{2\bar{X}}.$$

(b) (2 points). Let $X_1, \cdots, X_n$ be iid with common PDF

$$f_\theta(x) = \begin{cases} \frac{4x^3}{\theta^4}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise}. \end{cases}$$

Find the MLE of $\theta$.

**Solution:** The likelihood function is

$$\ell(\theta) = \frac{4^n}{\theta^{3n}} \prod_{i=1}^{n} X_i^3 I(0 \leq \max_{i \leq n}(X_i) \leq \theta).$$

Then, $\theta \geq \max_{i \leq n}(X_i)$. Thus, the MLE is $\max_{i \leq n}(X_i)$. 


4. (6 points). Let $X_1, \ldots, X_n$ be iid $N(0, \theta)$, where the PDF is

$$f_\theta(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}, -\infty \leq x \leq \infty, \theta > 0.$$ 

(a) (2 points). Find the MLE of $\theta$.

**Solution:** The loglikelihood function is

$$\ell(\theta) = \sum_{i=1}^{n} \log \left( \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{X_i^2}{2\theta}} \right) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \theta - \frac{1}{2\theta} \sum_{i=1}^{n} X_i^2.$$

Then,

$$\ell'(\theta) = \frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} X_i^2 \Rightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

(b) (2 points). Let $\theta_0$ be the true parameter. Compute the Fisher information of the MLE and provide the corresponding asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$.

**Solution:** By

$$\log f_\theta(x) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log \theta - \frac{x^2}{2\theta},$$

we have

$$\frac{\partial \log f_\theta(x)}{\partial \theta} = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2}$$

and

$$\frac{\partial^2 \log f_\theta(x)}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}.$$

Since $E(X^2) = \theta$, we have

$$I(\theta) = -E \left( \frac{1}{2\theta^2} - \frac{x^2}{\theta^3} \right) = \frac{1}{2\theta^2}.$$

Thus,

$$\sqrt{n}(\hat{\theta} - \theta_0) \overset{D}{\to} N(0, \theta^2).$$

(c) (2 points). Using the Delta theorem, provide the asymptotic distribution of $\sqrt{n}(\sqrt{\hat{\theta}} - \sqrt{\theta_0})$.

**Solution:** Let $g(\theta) = \sqrt{\theta}$. Then $g'(\theta) = 1/(2\sqrt{\theta})$. Thus,

$$\sqrt{n}[g(\hat{\theta}) - g(\theta_0)] \overset{D}{\to} N(0, [g'(\theta_0)]^2 \theta^2) = N(0, \theta^2/2).$$
5. (6 points). Let \( X_1, \ldots, X_n \) be iid \( \text{Uniform}[0, \theta] \), \( \theta > 0 \). Assume \( n \) is even and let \( m = n/2 \). Let \( \hat{\theta}_1 = \max_{i \leq n} (X_i) \) and \( \hat{\theta}_2 = \max_{i \leq m} (X_i) \).

(a) (2 points). Compute the bias of \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \), respectively.

\textbf{Solution:} The CDF of \( \max_{i \leq n} (X_i) \) is

\[ P(\max_{i \leq n} X_i \leq x) = \prod_{i=1}^{n} P(X_i \leq x) = \left( \frac{x}{\theta} \right)^n. \]

The PDF is \( nx^{n-1}/\theta^n \). Thus,

\[ E(\hat{\theta}_1) = \int_0^{\theta} \frac{nx^n}{\theta^n} dx = \frac{n\theta}{n+1}. \]

Thus,

\[ \text{Bias}(\hat{\theta}_1) = \frac{n\theta}{n+1} - \theta = \frac{\theta}{n+1}. \]

Similarly, we have

\[ \text{Bias}(\hat{\theta}_2) = \frac{\theta}{m+1} = \frac{2\theta^2}{n+2}. \]

(b) (2 points). Compute the variance of \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \), respectively.

\textbf{Solution:} By

\[ E(\hat{\theta}_1^2) = \int_0^{\theta} \frac{nx^{n+1}}{\theta^n} dx = \frac{n\theta^2}{n+2}. \]

we have

\[ V(\hat{\theta}_1) = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = \frac{n\theta^2}{(n+1)^2(n+2)}. \]

Similarly, we have

\[ V(\hat{\theta}_2) = \frac{m\theta^2}{(m+1)^2(m+2)} = \frac{8n\theta^2}{(2n+1)^2(2n+2)}. \]

(c) (2 points). Compare their MSEs. Explain which is a better estimator.

\textbf{Solution:} The MSE of \( \hat{\theta}_1 \) is

\[ \text{MSE}(\hat{\theta}_1) = \frac{\theta^2}{(n+1)^2} + \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}. \]

The MSE of \( \hat{\theta}_2 \) is

\[ \text{MSE}(\hat{\theta}_2) = \frac{8\theta^2}{(n+2)(n+4)}. \]

Their ratio is

\[ \frac{\text{MSE}(\hat{\theta}_1)}{\text{MSE}(\hat{\theta}_2)} = \frac{(n+2)(n+4)}{4(n+1)(n+2)} = \frac{n+4}{4n+1} < 1. \]

Thus, \( \hat{\theta}_1 \) is better than \( \hat{\theta}_2 \).