There are totally 42 points in the exam. The students with score higher than or equal to 40 points will receive 40 points. Please write down your name and student ID number below.

NAME: ________________________________
ID: ________________________________
1. (20 points) (1 point each). Fill in the blanks. Answers must be put in the blanks.

(a) (4 points). Let \( X_1 \) and \( X_2 \) be random variables with \( E(X_1) = 0.5 \), \( E(X_2) = 0.8 \), \( V(X_1) = 2.25 \), \( V(X_2) = 1.44 \), and \( \text{Cov}(X_1, X_2) = 0.9 \). Then, \( \text{Cor}(X_1, X_2) = \frac{0.9}{\sqrt{2.25 \times 1.44}} \). Let \( Y = X_1 - 1.5X_2 \). Then, \( E(Y) = \frac{0.5 - 1.5 	imes 0.8}{\sqrt{2.25 + 1.44 - 2 \times 0.9}} \), \( V(Y) = \frac{2.25 + 1.44 - 2 \times 0.9}{\sqrt{2.25 + 1.44 - 2 \times 0.9}} \), and \( \text{Cov}(X_1, Y) = \frac{0.9}{\sqrt{2.25 + 1.44 - 2 \times 0.9}} \).

(b) (3 points). Let \( X_1, \cdots, X_{10} \) be iid \( N(0, 4) \) and \( Y = \sum_{i=1}^{10} X_i^2 \). Then, the distribution of \( Y \) is \( \chi^2_{10} \), \( E(Y) = 10 \times 0 = 0 \), and \( V(Y) = 20 \times 0 = 0 \).

(c) (4 points). Let \( X_1, \cdots, X_{300} \) be iid with common PMF

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<td>( p )</td>
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Then, \( E(\bar{X}) = \frac{0.3 \times 0 + 0.4 \times 1 + 0.3 \times 2}{3} \), \( V(\bar{X}) = \frac{0.3 \times (0 - \frac{1}{3})^2 + 0.4 \times (1 - \frac{1}{3})^2 + 0.3 \times (2 - \frac{1}{3})^2}{3} \). By the central limit theorem, the approximate distribution of \( \bar{X} \) is \( N(\frac{1}{3}, \frac{0.333}{300}) \) and \( P(\bar{X} \leq 1.08) \approx \Phi(1.08). \)

(d) (3 points). Let \( X_1, \cdots, X_n \) be iid \( N(\mu, 1.4) \) with observed \( \bar{x} = 2.68 \). If \( n = 50 \), then the 99% confidence interval for \( \mu \) is \( \bar{x} \pm 2.576 \sqrt{\frac{1.4}{n}} \). If we want to test \( H_0 : \mu = 2 \) against \( H_0 : \mu \neq 2 \) at 0.01 significant level, then we reject \( H_0 \) based on the observations. If we want the 99% confidence interval to be less than 0.1, then we need \( n \) to be at least \( \frac{(z_{0.005})^2}{0.001^2} \).

(e) (3 points). Let \( X_1, \cdots, X_{10} \) be iid \( N(\mu, \sigma^2) \) with observed \( \bar{x} = 2.21 \) and \( s^2 = 1.59 \). Then, the 95% confidence interval for \( \mu \) is \( \bar{x} \pm 1.96 \sqrt{\frac{s^2}{10}} \), the 99% confidence interval for \( \mu \) is \( \bar{x} \pm 2.576 \sqrt{\frac{s^2}{10}} \), and the 95% confidence interval for \( \sigma^2 \) is \( \frac{s^2}{\chi^2_{0.025,9} = 19.03} \) and \( \frac{s^2}{\chi^2_{0.975,9} = 2.70} \).

(f) (3 points). Let \( X \sim Bin(n, \theta) \) with observed \( x = 270 \) when \( n = 600 \). Then, the maximum likelihood estimate of \( \theta \) is \( \frac{x}{n} \) and the 95% confidence interval for \( \theta \) is \( \frac{x}{n} \pm 1.96 \sqrt{\frac{\frac{x}{n} \times (1 - \frac{x}{n})}{n}} \). If we want the 95% confidence interval for \( \theta \) to be less than 0.01, then \( n \) should be at least \( \frac{\chi^2_{0.025,9}}{0.01^2} \).
2. (6 points). Let $X \sim \text{Bin}(15, \theta)$. Suppose we want to test

$$H_0 : \theta \leq 0.3 \leftrightarrow H_1 : \theta > 0.3.$$ 

Let the rejection region be $C = \{X \geq 8\}$. You need to use the binomial PMF given by Table D.6 from page 724 to 728 in the textbook. You need to show the main steps of your work.

(a) (2 points). Compute the type I error probability when $\theta = 0.25$.

(b) (2 points). Compute the type II error probability when $\theta = 0.4$.

(c) (2 points). Provide the significance level of the test.
3. (6 points). Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma_0^2)$, where $\sigma_0^2$ is known. Let $\hat{\mu}_1 = \frac{\sum_{i=1}^n X_i}{n-1}$ and $\hat{\mu}_2 = \frac{\sum_{i=1}^n X_i}{n+1}$, where $n \geq 2$.

(a) (2 points). Compute the biases of $\hat{\mu}_1$ and $\hat{\mu}_2$, respectively.

(b) (2 points). Compute the variances of $\hat{\mu}_1$ and $\hat{\mu}_2$, respectively.

(c) (2 points). Compare the MSEs of $\hat{\mu}_1$ and $\hat{\mu}_2$ and justify which one is better.
4. (6 points). Let $X_1, \cdots, X_n$ be iid with common PDF $f_\theta(x) = \theta^2 x e^{-\theta x}$ for $\theta, x > 0$. The common mean is $\mu = 2/\theta$ and common variance is $\sigma^2 = 2/\theta^2$.

(a) (2 points). Derive the MLE $\hat{\theta}$ of $\theta$.

(b) (2 points). Compute the Fisher information of the problem and use it to formula the approximate distribution of $\sqrt{n}(\hat{\theta} - \theta)$.

(c) (2 points). Using the Delta method to compute the approximate distribution of $\sqrt{n}(1/\hat{\theta} - 1/\theta)$. 
5. (4 points). Show your work in the computation of the p-value in the following problems.

(a) (2 points). Let $X_1, \cdots, X_{10}$ be iid $N(\mu, 2.5^2)$. Assume we want to test

$$H_0 : \mu \geq 2 \leftrightarrow H_1 : \mu < 2.$$ 

We have the observed value $\bar{x} = 0.908$.

(b) (2 points). Let $X \sim Bin(500, \theta)$. Suppose we want to test

$$H_0 : \mu \leq 0.4 \leftrightarrow H_1 : \mu > 0.4.$$ 

We have $X = 230$. 