QUALIFYING EXAM SOLUTIONS
STATISTICAL THEORY: Saturday, Jan 6, 2007, 8:00 am -12:00 pm

1. (a) The joint density is

$$f(x_1,\cdots,x_n,\theta)=\frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2\sigma^2}[\sum_{i=1}^n X_i^2-2\theta\sum_{i=1}^n X_i+n\theta^2]}\frac{1}{2a}e^{-\frac{|\theta|}{a}}.$$

Then, the marginal density of $x_1,\cdots,x_n$ is

$$\begin{aligned}&\int_{-\infty}^{\infty} f(x_1,\cdots,x_n,\theta)d\theta\\ =&\frac{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n X_i^2}}{2a(2\pi)^{n/2}}\{\int_{-\infty}^0 e^{-\frac{n}{2\sigma^2}[\theta^2-2\theta(\bar X+\frac{\sigma^2}{an})]}d\theta+\int_0^\infty e^{-\frac{n}{2\sigma^2}[\theta^2-2\theta(\bar X-\frac{\sigma^2}{an})]}d\theta\}\\ =&\frac{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n X_i^2}}{2a(2\pi)^{n/2}}\{e^{-\frac{n}{2\sigma^2}(\bar X+\frac{\sigma^2}{an})^2}\Phi(-\frac{\bar X+\sigma^2/(an)}{\sigma/\sqrt{n}})+e^{-\frac{n}{2\sigma^2}(\bar X-\frac{\sigma^2}{an})^2}\Phi(\frac{\bar X-\sigma^2/(an)}{\sigma/\sqrt{n}})\}\end{aligned}$$

Then, the posterior density of θ is

$$\begin{aligned}g(\theta|x_1,\cdots,x_n)=&e^{-\frac{n}{2\sigma^2}[\theta^2-2\theta\bar X+\frac{|\theta|}{a}]}\\ &\times\{e^{-\frac{n}{2\sigma^2}(\bar X+\frac{\sigma^2}{an})^2}\Phi(-\frac{\bar X+\sigma^2/(an)}{\sigma/\sqrt{n}})+e^{-\frac{n}{2\sigma^2}(\bar X-\frac{\sigma^2}{an})^2}\Phi(\frac{\bar X-\sigma^2/(an)}{\sigma/\sqrt{n}})\}^{-1}\end{aligned}$$

and the posterior probability of $\theta>K$ is

$$P_{\bar X}(\theta>K)=\int_K^\infty g(\theta|x_1,\cdots,x_n)d\theta$$

$$=\begin{cases}\dfrac{e^{-\frac{n}{2\sigma^2}(\bar X+\frac{\sigma^2}{an})^2}[\Phi(-\frac{[\bar X+\sigma^2/(an)]}{\sigma/\sqrt{n}})-\Phi(\frac{K-[\bar X+\sigma^2/(an)]}{\sigma/\sqrt{n}})]+e^{-\frac{n}{2\sigma^2}(\bar X-\frac{\sigma^2}{an})^2}\Phi(\frac{\bar X-\sigma^2/(an)}{\sigma/\sqrt{n}})}{e^{-\frac{n}{2\sigma^2}(\bar X+\frac{\sigma^2}{an})^2}\Phi(-\frac{\bar X+\sigma^2/(an)}{\sigma/\sqrt{n}})+e^{-\frac{n}{2\sigma^2}(\bar X-\frac{\sigma^2}{an})^2}\Phi(\frac{\bar X-\sigma^2/(an)}{\sigma/\sqrt{n}})} & \text{when } K<0\\ \dfrac{e^{-\frac{n}{2\sigma^2}(\bar X-\frac{\sigma^2}{an})^2}\Phi(\frac{\bar X-K-\sigma^2/(an)}{\sigma/\sqrt{n}})}{e^{-\frac{n}{2\sigma^2}(\bar X+\frac{\sigma^2}{an})^2}\Phi(-\frac{\bar X+\sigma^2/(an)}{\sigma/\sqrt{n}})+e^{-\frac{n}{2\sigma^2}(\bar X-\frac{\sigma^2}{an})^2}\Phi(\frac{\bar X-\sigma^2/(an)}{\sigma/\sqrt{n}})} & \text{when } K\ge 0\end{cases}$$

(b) As $a\to\infty$, $\sigma^2/(an)\to 0$ and we have

$$P_{\bar X}(\theta>K)\to\Phi(\frac{\bar X-K}{\sigma/\sqrt{n}}).$$

(c) It is clear that the posterrior probability goes to the p-value of the classical hypothesis test.

2. (a) Note that $E(X_i)=0$ and $E(X_i^2)=(1-p)+pv^2$. Thus, an ME can be

$$\hat p=\frac{\frac{1}{n}\sum_{i=1}^n X_i^2-1}{v^2-1}.$$

By SLLN, all moment estimators are consistent. Therefore, the above estimator is consistent.

(b) Note that $E(X_i^4) = 3(1-p)+3pv^4$. Then, we can estimate $3(1-p)+3pv^4$ by $\sum_{i=1}^n X_i^4$. It gives us the following estimator of p when $v>1$ is unknown by solving

$$\hat{p}(\hat{v}^2-1) = \frac{1}{n}\sum_{i=1}^n X_i^2 - 1$$

and

$$\hat{p}(\hat{v}^4-1) = \frac{1}{3n}\sum_{i=1}^n X_i^4 - 1.$$

Then, we have

$$\hat{v}^2 = \frac{\frac{1}{3n}\sum_{i=1}^n X_i^4 - 1}{\frac{1}{n}\sum_{i=1}^n X_i^2 - 1} - 1$$

and

$$\hat{p} = \frac{(\frac{1}{n}\sum_{i=1}^n X_i^2 - 1)^2}{\frac{1}{3n}\sum_{i=1}^n X_i^4 - \frac{2}{n}\sum_{i=1}^n X_i^2 + 1}$$

3. (a) The density function is

$$f(x) = I_{[\theta,\theta+1]}(x) = I_{(-\infty,\theta+1]}(x)I_{[\theta,\infty)}(x)$$

Then, the joint density is

$$f(x_1,\cdots,x_n) = \prod_{i=1}^n I_{[\theta,\theta+1]}(X_i) = I_{(-\infty,\theta+1]}(\max X_i)I_{[\theta,\infty)}(\min X_i).$$

Thus, the minimal sufficient statistic is $(\min X_i, \max X_i)$.

(b) (i) Look at the likelihood ratio, we have

$$\frac{f_{H_0}(x_1,\cdots,x_n)}{f_{H_1}(x_1,\cdots,x_n)} = \begin{cases} \infty & \text{when } \max(x_i) < 1 \\ 0 & \text{when } \min(x_i) > 1 \end{cases}$$

Thus, the UMP test rejects H_0 if $\min(x_i) > 1$. (ii) Still look at the likelihood ratio, we have

$$\frac{f_{H_0}(x_1,\cdots,x_n)}{f_{H_1}(x_1,\cdots,x_n)} = \begin{cases} \infty & \text{when } \min(x_i) < \theta_1 \\ 0 & \text{when } \max(x_i) > 1 \\ 1 & \text{when } \theta_1 \le \min(x_i) \le 1 \end{cases}$$

Thus, the MP test rejects H_0 is $\max(x_i) > 1$ and accepts H_0 if $\min(x_i) < \theta_1$. Note that under H_0, the CDF of $X_{(1)} = \min(X_i)$ is

$$F_{X_{(1)}}(x) = 1-(1-x)^n.$$

Thus, if $(1-\theta_1)^n \le \alpha$, then the MP test reject H_0 when $\min(X_i) \ge \theta_1$; if $(1-\theta_1)^n > \alpha$, then the MP test rejects H_0 with probability $\alpha/(1-\theta_1)^n$ when $\min(X_i) \ge \theta_1$.

4. (a) Let $B(x;n,p)$ be the binomial CDF. Note that

$$\frac{(n+1)!}{j!(n+1-j)!} = \frac{n!}{j!(n-j)!} + \frac{n!}{(j-1)![n-(j-1)]!}$$

We have

$$
\begin{aligned}
&B(x;n+1,p)\\
=&(1-p)B(x;n,p)+pB(x-1;n,p)\\
\leq&(1-p)B(x;n,p)+pB(x;n,p)\\
=&B(x;n,p).
\end{aligned}
$$

Thus, $B(x;n,p)$ is nonincreaseing in n and so $\mathcal{N}_U$ consists of a segment of consecutive integers.

(b) Straightforwardly, we have

$$
\begin{aligned}
P\{n\in\mathcal{N}_u\}=&P\{B(x-1;n,p)\leq U<B(x;n,p)\}\\
=&B(x;n,p)-B(x-1;n,p)\\
=&\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}.
\end{aligned}
$$

5. Suppose $\delta(X)$ is an estimator of θ and let $f(x)$ be the density of X, denote $L(\delta(x),\theta)$ as the loss function and $R(\delta(X),\theta)=\int_{\mathcal{X}}L(\delta(x),\theta)P_\theta(dx)$ as the risk function.

(a) True. Suppose δ_M be the minimax rule. Then, the minimax rule satisfies

$$\max_\theta R(\delta_M,\theta)=\min_\delta\max_\theta R(\delta,\theta)$$

Let $\pi(\theta)$ be a prior distribution. Then, the Bayes risk is

$$\int R(\delta_M,\theta)\pi(d\theta)\leq\int\max_\theta R(\delta_M,\theta)\pi(d\theta)=\max_\theta R(\delta_M,\theta).$$

Thus, the Bayes risk of minimax rule is never greater than the minimax risk

(b) False. Suppose θ is the median and $L(\delta,\theta)=|\delta-\theta|$. If $n=2m$ is even number, then any rule choosing number between $X_{(m)}$ and $X_{(m+1)}$ is a Bayes rule. It is known that this is admissible.