QUALIFYING EXAM SOLUTIONS
STATISTICAL THEORY
Saturday, Aug 11, 2007, 8:00 am -12:00 pm

1. (a) The CDF of $Y = X_{(n)}$ is

$$G(y|\theta) = G_{X_{(n)}}(y|\theta) = (\frac{C}{\theta})^n$$

and the PDF is

$$g(y|\theta) = g_{X_{(n)}}(y|\theta) = \frac{ny^{n-1}}{\theta^n},\ 0 \leq y \leq \theta,\ \theta > 0.$$

For a test $H_0 : \theta = 1$ versus $H_1 : \theta > 0$, we have the likelihood ratio

$$\frac{g(y|\theta)}{g(y|1)} = \begin{cases} \theta^{-n}, & \text{when } 0 < y \leq 1 \\ \infty & \text{when } y > 1. \end{cases}$$

If the test rejects H_0 if $Y > 0$, then we have

$$P_{\theta=1}[Y > C] = 1 - C^n = \alpha \Rightarrow C = (1-\alpha)^{1/n}.$$

If $\alpha = 0.05$, then we have

$$C = 0.95^{1/n}.$$

(b) The power function is

$$\beta(\theta) = P_\theta(Y \geq C) = 1 - (\frac{(1-\alpha)^{1/n}}{\theta})^n = 1 - \frac{1-\alpha}{\theta^n}.$$

If $\alpha = 0.05$, it is

$$\beta(\theta) = 1 - \frac{0.95}{\theta^n}.$$

(c) We need

$$1 - \beta(1.5) = \frac{0.95}{1.5^n} \leq 0.05 \Rightarrow n \geq \frac{\log(0.95/0.05)}{\log(1.5)} = 7.26.$$

Thus, n is at least 8.

2. (a) In this case, we have $\sum_{i=1}^n C_i = 1$. Then, we have

$$MSE = E[\sum_{i=1}^n C_i(Y_i - \mu)]^2 = \sum_{i=1}^n C_i^2 \geq \frac{1}{n}$$

with equality hold if and only if $C_i = 1/n$. Thus, the optimal is $\hat{Y} = \bar{Y}$ and $MSE = 1/n$.

(b) In this case, $E(\hat{Y}) = \mu \sum_{i=1}^n C_i$. Thus, the estimate is baised with bais

$$\mu\delta = \mu(\sum_{i=1}^n C_i - 1).$$

Then, we have

$$MSE = E[\sum_{i=1}^n C_i(Y_i - \mu) + \delta\mu]^2 = E[\sum_{i=1}^n C_i(Y_i - \mu)]^2 + \delta^2\mu^2 = \sum_{i=1}^n C_i^2 + \delta^2\mu^2.$$

Note that

$$\sum_{i=1}^{n} C_i^2 \geq \frac{1}{n}(\sum_{i=1}^{n} C_i)^2 = \frac{(\delta+1)^2}{n}$$

with equality hold if and only if $C_1 = C_2 = \cdots = C_n$. We have

$$MSE \geq (\mu^2 + \frac{1}{n})\delta^2 + \frac{2\delta}{n} + \frac{1}{n} \geq \frac{\mu^2}{1+n\mu^2}$$

with equality holds if and only if $\delta = -1/(1+n\mu^2)$ and $C_1 = C_2 = \cdots = C_n$ which is equivalent to $C_1 = C_2 = \cdots = C_n = \mu^2/(1+\mu^2)$. Thus, the best $\hat{Y}$ is

$$\hat{Y}_{best} = \frac{\mu_0^2}{1+n\mu_0^2}\sum_{i=1}^{n} Y_i$$

and the optimal MSE is

$$MSE_{optimal} = \frac{\mu_0^2}{1+n\mu_0^2}.$$

(c) Following the same idea of part b, we can define

$$\delta = C_0 + (\sum_{i=1}^{n} C_i - 1)\mu_0 = C_0 + (nC-1)\mu_0 = (C_0 - \mu_0) + nC.$$

The best case also takes $C_1 = \cdots = C_n = C$. Then, we have

$$\begin{aligned} MSE =& E[(\sum_{i=1}^{n} C_i(Y_i - \mu))^2] + \delta^2 \\ =& nC^2 + \delta^2 \\ =& (n^2+n)C^2 + 2n(C_0-\mu_0)C + (C_0-\mu_0)^2. \end{aligned}$$

The best C and C_0 can be solved by taking partial derivatives as

$$\frac{\partial MSE}{\partial C} = (n+1)C + (C_0 - \mu_0) = 0$$

and

$$\frac{\partial MSE}{\partial C_0} = (C_0 - \mu_0) + C = 0.$$

Then, we have $C = 0$ and $C_0 = \mu_0$. In this case, the best $\hat{Y} = \mu_0$ which gives $MSE = 0$.

3. (a) It is very clear that the indicator function is unbiased since

$$E(I_{X_1>2}) = P(X_1 > 2) = e^{-2/\theta}.$$

(b) We first look at the likelihood function

$$L = \prod_{i=1}^{n}(\frac{1}{\theta})e^{-\frac{x_i}{\theta}} I_{\theta>0} I_{x_i>0} = (\theta^{-n} I_{\theta>0})(I_{\min x_i>0})e^{-\frac{1}{\theta}\sum_{i=1}^{n} X_i}.$$

Then from the factorization theorem, we have $S = \sum_{i=1}^{n} X_i$ is the CSS. Note that this is the exponential density, which is also the $Gamma(1, \theta)$ density. Thus S follows $Gamma(n, \theta)$ distribution which has the density

$$g(s|\theta) = g_S(s|\theta) = \frac{s^{n-1}}{\theta^n} e^{-s/\theta}, \ s > 0, \ \theta > 0.$$

Note that X_1 and $X_2 + \cdots + X_n$ independently following $\Gamma(1, \theta)$ and $\Gamma(n-1, \theta)$ respectively. We have given S, X_1/S follows $Beta(1/n, (n-1)/n)$ distribution. Thus, the UMVUE can be derived by

$$E(I_{X_1 > 2}|S) = 1 - P(X_1 < 2|S) = 1 - P(\frac{X_i}{S} < \frac{2}{S}|S) = 1 - B_{\frac{1}{n}, \frac{n-1}{n}}(\frac{2}{S}),$$

where $B_{a,b}(\cdot)$ is the CDF of Beta distribution with parameters a and b respectively.

(c) This cannot be unbiased since it is not the UMVUE when $n = 1$.

4. (a) It is enough to look at the CSS $T = \sum_{i=1}^{n} Y_i$ which follows $Poisso(n\theta)$. The joint PMF-PDF of T and θ is

$$f(t, \theta) = \frac{(n\theta)^t}{t!} e^{-n\theta} \frac{\lambda^p}{\Gamma(p)} \theta^{p-1} e^{-\lambda\theta}.$$

The PMF of T is

$$\bar{f}(t) = \int_0^\infty f(t, \theta) d\theta = \frac{\Gamma(t+p)}{t!\Gamma(p)} \frac{n^t}{(n+\lambda)^{t+p}}.$$

Then, the posterior density of θ given T is

$$q(\theta|t) = \frac{(n+\lambda)^{t+p}}{\Gamma(t+p)} \theta^{t+p-1} e^{-(n+\lambda)\theta}.$$

The posterior loss is

$$E_q[\frac{(\theta - a)^2}{\theta}] = \frac{(n+\lambda)^{t+p}}{\Gamma(t+p)} \int_0^\infty (\theta - a)^2 \theta^{t+p-2} e^{(n+\lambda)\theta} d\theta$$

which is minimized by

$$a = \frac{t+p-1}{n+\lambda}.$$

Therefore, the Bayes estimator is

$$\delta_\pi = \frac{t+p-1}{n+\lambda}.$$

(b) The risk is

$$\begin{aligned} R(\theta, \delta_\pi) =& E[\frac{(\theta - \frac{t+p-1}{n+\lambda})^2}{\theta}] = \frac{1}{\theta}[\frac{n\theta}{(n+\lambda)^2} + (\frac{\lambda\theta - p + 1}{n+\lambda})^2] \\ =& \frac{n - 2\lambda(p-1)}{(n+\lambda)^2} + \frac{\lambda^2\theta}{(n+\lambda)^2} + \frac{(p-1)}{\theta(n+\lambda)^2}. \end{aligned}$$

Thus, the Bayes risk is

$$r_\pi(\delta_\pi) = E_\pi[R(\theta, \delta_\pi)] = \frac{n - 2\lambda(p-1)}{(n+\lambda)^2} + \frac{\lambda p}{(n+\lambda)^2} + \frac{\lambda}{(n+\lambda)^2} = \frac{n - \lambda p + 3\lambda}{(n+\lambda)^2}.$$

5. The density is

$$f_{a,b,c} = \frac{1}{bc} I_{(0,c)}(x) I_{(ax,ax+b)}(y) = \frac{1}{bc} I_{(0,c)}(x) I_{(0,b)}(y - ax).$$

The likeihood function is

$$L = \frac{1}{(bc)^n} \prod_{i=1}^{n} I_{(0,c)}(x_i) I_{(0,b)}(y_i - ax_i).$$

To maximize L, we need small b and c. Thus, we have $c = \max(x_1, \cdots, x_n)$. Next, we compute the MLE of b and a. Note that when a is given, the MLE of b is

$$\hat{b}_a = \max_{\{(x_i, y_i): i=1, \cdots, n\}} (y_i - ax_i).$$

Thus, the MLE $\hat{a}$ of a minimizes $\hat{b}_a$ and $\hat{b} = \hat{b}_{\hat{a}}$. Thus, we have

$$\hat{a} = \arg\min_a \max_{\{(x_i, y_i): i=1, \cdots, n\}} (y_i - ax_i)$$

and

$$\hat{b} = \max_{\{(x_i, y_i): i=1, \cdots, n\}} (y_i - \hat{a}x_i).$$