QUALIFYING EXAM SOLUTIONS

STATISTICAL THEORY: Thursday, Jan 5, 2006, 8:00 am -12:00 pm

1. (a) The conditional distribution of Y given ϕ is $N(0, 1/\phi)$ by property of bivariate normal. The marginal density of Y is

$$\begin{aligned} p_Y(y) &= \int_0^\infty f(y|\phi)g(\phi)d\phi \\ &\propto \int_0^\infty \sqrt{\phi}e^{-\phi y^2/2}\phi e^{-2\phi}d\phi \\ &= \int_0^\infty \phi^{3/2}e^{-(1+y^2/4)2\phi}d\phi \\ &\propto (1+y^2/4)^{-(4+1)/2} \end{aligned}$$

which is t distribution with df 5.

(b) $E(X) = E(E(X|\phi)) = 0$, $E(Y) = E(E(Y|\phi)) = 0$. Also with $\rho = 0$, we have

$$cov(X,Y) = E(XY) = E(E(XY|\phi)) = E(E(X|\phi)E(Y|\phi)) = 0.$$

Therefore X and Y are uncorrelated. On the other hand, the joint density

$$\begin{aligned} p_{X,Y}(x,y) &\propto \int_0^\infty \phi e^{-\phi(x^2+y^2)/2}\phi e^{-2\phi}d\phi \\ &= \int_0^\infty \phi^2 e^{-((x^2+y^2)/2+2)\phi}d\phi \\ &\propto (4+x^2+y^2)^{-3} \end{aligned}$$

which implies dependence.

2. (a) $X_{(n)} = \max(X_i)$ is the complete sufficient statistic. Sufficiency is by the factorization theorem. Completeness is easy to verify.

(b) Note $E(\overline{X}) = E(X_1) = \sum_{x=1}^N x/N = (1+N)/2$, therefore $2\overline{X} - 1$ is an unbiased estimate of N. By the Rao-Blackwell theorem and the property of the complete sufficient statistic, the minimum variance unbiased estimate of N is $2E(\overline{X}|X_{(n)}) - 1$.

3. (a) If τ^2 is known, the Bayes estimator, for the given sum of squared errors loss, of μ is the posterior mean. Note that the posterior distribution of μ_i is $N(\frac{\tau^2}{\tau^2+1}X_i, \frac{\tau^2}{\tau^2+1})$, independently for $i = 1, ..., p$. Therefore the Bayes estimator of $\mu = (\mu_1, ..., \mu_p)$ is $\delta(\mathbf{X}) = (\frac{\tau^2}{\tau^2+1}X_1, ..., \frac{\tau^2}{\tau^2+1}X_p)$. The Bayes risk is given by

$$\sum_{i=1}^p var(\mu_i|X_i) = \frac{p\tau^2}{\tau^2+1}.$$

(b) Marginally the randmo variables X_i are iid $N(0, \tau^2+1)$, so that $X_i/\sqrt{\tau^2+1} \sim N(0,1)$ and $||\mathbf{X}||^2/(\tau^2+1) \sim \chi^2_p$. By hint,

$$E\left[1 - \frac{p-2}{||\mathbf{X}||^2}\right] = \frac{\tau^2}{\tau^2+1}.$$

Replacing $\tau^2/(\tau^2+1)$ by $1-(p-2)/||\mathbf{X}||^2$ yields the James-Stein estimator.

4. (a) Define $t(x) = \max(x_1, ..., x_n)$. Then for $0 < \theta_1 < \theta_2$,

$$\frac{f(\mathbf{x};\theta_2)}{f(\mathbf{x};\theta_1)} = \begin{cases} \left(\frac{\theta_1}{\theta_2}\right)^n & 0 < t(x) \le \theta_1 \\ \infty & \theta_1 < t(x) \le \theta_2 \end{cases}$$

This is a non-decreasing function of $t(x)$, i.e. the uniform distribution is a MLR family. Therefore the UMP test exists with the form

$$\varphi(x) = \begin{cases} 1 & t(x) > t_0 \\ 0 & t(x) < t_0 \end{cases}$$

where t_0 satisfies $P_{\theta_0}(T > t_0) = \alpha$ i.e. $t_0 = \theta_0(1-\alpha)^{1/n}$.

(b) It is easy to verify that $E_{\theta_0}\phi(X) = \alpha$. $\forall \delta(X)$ of size α, $E_{\theta_0}\delta(X) = \alpha$. Consider any $\theta > \theta_0$. Let $k = (\theta_0/\theta)^n$. Following the idea of Nayman-Pearson Lemma, we can show

$$E_\theta(\phi(X) - \delta(X)) - kE_{\theta_0}(\phi(X) - \delta(X)) \ge 0$$

That is $E_\theta(\phi(X) - \delta(X)) \ge 0$. Similarly, for $\theta < \theta_0$, the inequality holds as well. Therefore $E_\theta(\phi(X)) \ge E_\theta(\delta(X))$ for all $\theta \ne \theta_0$, and $\phi(X)$ is the UMP test.

5. (a) The log-likelihood function is

$$\ell(\theta) = \sum_{i=1}^n \log[f(x_i;\theta)] = n\log(\theta+1) + \theta\sum_{i=1}^n \log(x_i).$$

Then, we have

$$\ell'(\theta) = \frac{n}{\theta+1} + \sum_{i=1}^n \log(x_i) \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^n \log(x_i)} - 1.$$

The fisher information matrix is

$$\ell''(\theta) = -\frac{n}{(\theta+1)^2}.$$

Therefore, the asymptotic distribution of the MLE is

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{L} N(0, (\theta+1)^2).$$

(b) Let the variance stabilizing transformation be $g(\theta)$. Then, we have

$$\sqrt{n}[g(\hat{\theta}) - g(\theta)] \xrightarrow{L} N(0, [g'(\theta)]^2(\theta+1)^2).$$

Set

$$[g'(\theta)]^2(\theta+1)^2 = 1 \Rightarrow g'(\theta) = \frac{1}{\theta+1} \Rightarrow g(\theta) = \log(1+\theta).$$

Therefore, we have

$$\sqrt{n}[\log(1+\hat{\theta}) - \log(1+\theta)] \xrightarrow{L} N(0,1).$$

(c) The 95% asymptotic confidence interval is

$$\begin{aligned}
& -1.96 \le \sqrt{n}[\log(1+\hat{\theta}) - \log(1+\theta)] \le 1.96 \\
\Rightarrow & \log(1+\hat{\theta}) - \frac{1.96}{\sqrt{n}} \le \log(1+\theta) \le \log(1+\hat{\theta}) + \frac{1.96}{\sqrt{n}} \\
\Rightarrow & (1+\hat{\theta})e^{-1.96/\sqrt{n}} \le 1+\theta \le (1+\hat{\theta})e^{1.96/\sqrt{n}} \\
\Rightarrow & (1+\hat{\theta})e^{-1.96/\sqrt{n}} - 1 \le \theta \le (1+\hat{\theta})e^{1.96/\sqrt{n}} - 1.
\end{aligned}$$

The 95% asymptotic confidence interval is

$$[(1+\hat{\theta})e^{-1.96/\sqrt{n}} - 1, (1+\hat{\theta})e^{1.96/\sqrt{n}} - 1].$$

6.

$$\begin{aligned}
P(Z^2 \le a) =& P(-\sqrt{a} \le Z \le \sqrt{a}) \\
=& P(Z \le \sqrt{a}) - P(Z \le -\sqrt{a}) \\
=& P(Z > -\sqrt{a}) - P(Z > \sqrt{a}) \\
=& P(X > -\sqrt{a})P(Y > -\sqrt{a}) - P(X > \sqrt{a})P(Y > \sqrt{a}) \\
=& [1 - F(-\sqrt{a})]^2 - [1 - F(\sqrt{a})]^2 \\
=& F(\sqrt{a})^2 - [1 - F(\sqrt{a})]^2 \\
=& 2F(\sqrt{a}) - 1
\end{aligned}$$

If we differentiate both sides with respect to a, LHS=pdf of Z^2.

$$\begin{aligned}
RHS =& \frac{d}{da}(2F(\sqrt{a})) \\
=& \frac{\phi(\sqrt{a})}{\sqrt{a}} \\
=& \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{a}}e^{-a/2} \\
=& \frac{1}{\gamma(1/2)2^{1/2}}a^{1/2-1}e^{-a/2} \\
=& Gamma(1/2, 2) = \chi_1^2
\end{aligned}$$

7. (a) For the first part, write:

$$R^2 = 1 - \frac{(Y - \hat{Y}, Y - \hat{Y})}{(Y, Y)} = \frac{(Y, \hat{Y})}{(Y, Y)} = \frac{(\hat{Y}, \hat{Y})}{(Y, Y)},$$

the claim follows directly by noticing $\hat{Y} = \frac{(X,Y)}{(X,X)} \cdot X$.

(b) For the second part, denote $e = e_{Z;X}$ and $d = e_{Y;X}$ for short. Noticing both e and d are centered, all we need to show is:

$$R^2_{ZY|X} = \frac{(e, d)^2}{(d, d)(e, e)}.$$

Clearly, $Y = \frac{(Y,X)}{(X,X)} X + d$. We can thus write the model as:

$$Z = (\beta_1 + \beta_2 \cdot \frac{(Y, X)}{(X, X)}) X + d,$$

noticing $X \perp d$. As a result, $\hat{Z} = \frac{(Z,X)}{(X,X)} X + \frac{(Z,d)}{(d,d)} d$, and

$$SSE(X, Y) = \|Z - \hat{Z}\|^2 = (Z, Z) - \frac{(Z, X)^2}{(X, X)} - \frac{(Z, d)^2}{(d, d)}.$$

At the same time, consider another model $Z = \beta X + \epsilon$, we have:

$$Z = \hat{Z} + e, \qquad \hat{Z} \equiv \frac{(Z, X)}{(X, X)} X,$$

thus

$$SSE(X) = \|Z - \hat{Z}\|^2 = (Z, Z) - \frac{(Z, X)^2}{(X, X)} \equiv (e, e),$$

combining these gives:

$$R_{ZY|X} = \frac{(Z, d)}{(d, d)(e, e)} = \frac{(e, d)^2}{(d, d)(e, e)},$$

where the last equality uses the fact that $Z = \frac{(Z,X)}{(X,X)} X + e$ and $X \perp d$. This finishes the proof. $\diamondsuit$

8. Let $X = PDQ$ be the singular decomposition of X, where P and Q are n by p and p by p orthogonal matrix respectively so that $P'P = I_p$ and $Q'Q = I_p$, D is a diagonal matrix with diagonal entries $d_1 \geq d_2 \geq \ldots d_p \geq 0$ being the singular values of X. By the singular decomposition and direct calculations, we have $(X'X + \lambda I)^{-1} \cdot (X'X) \cdot (X'X + \lambda I)^{-1} = Q' D_\lambda Q$, where D_λ is diagonal matrix whose diagonal entries are $d_i^2/(d_i^2 + \lambda)^2$ for $i = 1, 2, \ldots, p$. Now if we write the k-th column of Q by $(a_1, a_2, \ldots, a_p)'$, then the k-th diagonal term of $(X'X + \lambda I)^{-1} \cdot (X'X) \cdot (X'X + \lambda I)^{-1}$ equals to:

$$\sum_{i=1}^{p} \frac{d_i^2}{(d_i^2 + \lambda)^2} a_i^2,$$

and the claim follows directly. $\diamondsuit$