QUALIFYING EXAM SOLUTIONS
STATISTICAL THEORY: Saturday, Aug 12, 2006, 8:00 am -12:00 pm

1. Assume h is strictly increasing.

   (a) This is not invariant. Suppose $X \sim U[0,\theta]$ for $\theta > 0$. Assume $n > 5$. Let prior be uniform. Then, the posterior density of θ given $X_{(n)}$ is

   $$q(\theta|x_{(n)} = x) = \frac{n\theta^{-n}x^{n-1}}{\int_x^\infty n\theta^{-n}x^{n-1}d\theta} = \frac{(n-1)x^{n-1}}{\theta^n}$$

   when $\theta \geq x_{(n)}$. The posterior mean is

   $$\hat{\theta} = E(\theta|x_{(n)}) = (n-1)x^{n-1}\int_x^\infty \frac{1}{\theta^{n-1}}d\theta = \frac{n-1}{n-2}x.$$

   However, if we choose $\eta = \theta^2$. Then, the posterior density of η is

   $$q_1(\eta|x_{(n)} = x) = \frac{(n-1)x^{n-1}}{\eta^{n/2}}\frac{d\theta}{d\eta} = \frac{(n-1)x^{n-1}}{2\eta^{(n+1)/2}}$$

   and the posterior mean is

   $$\hat{\eta} = \frac{(n-1)x^{n-1}}{2}\int_{x^2}^\infty \frac{1}{\eta^{(n-1)/2}}d\eta == \frac{n-1}{n-3}x^2 \neq (\frac{n-1}{n-2}x)^2.$$

   (b) It is invariant. Proof is easy. Suppose the posterior distribution of θ is $Q(\theta|x_1,\cdots,x_n)$. Then, the posterior density of η is $Q(h^{-1}(\eta)|x_1,\cdots,x_n)$ and the posterior median satisfies

   $$Q(h^{-1}(\eta) \leq h^{-1}(\hat{\eta}_M)|x_1,\cdots,x_n) = 1/2 \Rightarrow Q(\theta \leq h^{-1}(\eta_M)|x_1,\cdots,x_n) = 1/2.$$

   Thus, $\hat{\eta} = h(\hat{\theta}_M)$.

   (c) It is invariant. Look at the likelihood function

   $$\ell(\theta|x_1,\cdots,x_n) = \ell(h^{-1}(\eta)|x_1,\cdots,x_n).$$

   If $\hat{\theta}$ maximizes the likelihood function of θ, then $h^{-1}(h(\hat{\theta}))$ also maximizes the likeihood function of η.

2. (a) The density of X_i is

   $$f(x) = \theta e^{-\theta x}$$

   for $x > 0$. The loglikelihood function is

   $$\ell(\theta|x_1,\cdots,x_n) = n\log\theta - \theta\sum_{i=1}^n x_i.$$

   Then,

   $$\ell'(\theta|x_1,\cdots,x_n) = \frac{n}{\theta} - n\bar{x} \Rightarrow \hat{\theta} = \frac{1}{\bar{X}}.$$

Note that $Exp(\theta)$ distribution is $Gamma(1, \theta)$. Then, $\sum_{i=1}^n X_i$ follows $Gamma(n, \theta)$ distribution and so $\bar{X} \sim Gamma(n, n\theta)$. Then, the density of $\bar{X}$ is

$$f_{\bar{X}}(x) = \frac{n^n \theta^n}{\Gamma(n)} x^{n-1} e^{-n\theta x}.$$

Then,

$$E(\frac{1}{\bar{X}}) = \int_0^\infty \frac{n^n \theta^n}{\Gamma(n)} x^{n-2} e^{-n\theta x} dx = \frac{n\theta}{n-1}.$$

Thus it is biased but $\frac{n-1}{n}\hat{\theta}$ is unbiased.

(b) Since

$$E[(\frac{d \log(\prod_{i=1}^n \theta e^{-\theta X_i})}{d\theta})^2] = E[(\frac{n}{\theta} - \sum_{i=1}^n X_i)^2] = \frac{n}{\theta^2}.$$

the Cramer-Rao Lower bound is θ^2/n. Now, consider

$$E[(\frac{n-1}{n}\hat{\theta})^2] = \frac{(n-1)^2}{n^2} \int_0^\infty \frac{n^n \theta^n}{\Gamma(n)} x^{n-3} e^{-n\theta x} dx = \frac{(n-1)\theta^2}{n-2}.$$

Thus,

$$V(\frac{n-1}{n}\hat{\theta}) = \frac{(n-1)\theta^2}{n-2} - \theta^2 = \frac{\theta^2}{n-2}.$$

Since the above is greater than the CR lower bound and this is the UMVUE, the CR lower bound can not be obtained.

3. (a) Straightforwardly, we have

$$\begin{aligned}\frac{1-\theta^n-(1-\theta)^n}{\theta(1-\theta)} &= \frac{(1+\theta+\cdots+\theta^{n-1})-(1-\theta)^{n-1}}{\theta} \\ &= (1+\theta+\cdots+\theta^{n-2}) + [1+(1-\theta)+(1-\theta)^2+\cdots+(1-\theta)^{n-2}].\end{aligned}$$

Thus, it is proper if $n \geq 2$ and

$$\int_0^1 \pi(\theta) d\theta = 2(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}).$$

The marginal density of X is

$$\int_0^1 \binom{n}{r} \theta^{r-1}(1-\theta)^{n-r-1} d\theta = \frac{n}{r(n-r)}.$$

Thus, the posterior density is

$$q(\theta|r) = \frac{(n-1)!}{(r-1)!(n-r-1)!} \theta^{r-1}(1-\theta)^{n-r-1}$$

which is the density of $Beta(r, n-r)$. Thus the posterior mean is r/n.

(b) Since x/n is a Bayesian estimator under the square error loss, it is admissible and thus there is not such an estimator.

4. (a) Note that

$$P(X_i = s, Y_i = t) = \eta_{s|t}\lambda_t$$

and

$$P(X_i = s) = \sum_{t=1}^{k} \eta_{s|t}\lambda_t$$

Thus, the loglikelihood function is

$$\ell(\theta|D_1 \cup D_2) = \sum_{i=1}^{n}(\log \eta_{X_i|Y_i} + \log \lambda_{Y_i}) + \sum_{i=n+1}^{n+m} \log(\sum_{t=1}^{k} \eta_{X_i|t}\lambda_t).$$

(b) We choose uniform $q(y)$ and then

$$\log(\frac{\sum_{t=1}^{k} \eta_{X_i|t}\lambda_t}{k}) \geq \frac{1}{k}\sum_{t=1}^{k}(\log \eta_{X_i|t} + \log \lambda_t).$$

Thus, the lower bound of $\ell(\theta|D_1 \cup D_2)$ is

$$\sum_{i=1}^{n}(\log \eta_{X_i|Y_i} + \log \lambda_{Y_i}) + \frac{1}{k}\sum_{i=n+1}^{n+m}\sum_{t=1}^{k}(\log \eta_{X_i|t} + \log \lambda_t) - m\log(k).$$

(c)

$$P(y = t|x = s) = \frac{\eta_{s|t}\lambda_t}{\sum_{t=1}^{k} \eta_{s|t}\lambda_t}.$$

Then, we have

$$\begin{aligned}
&E_q[\log(\frac{\eta_{X_i|t}\lambda_t / \sum_{j=1}^{k} \eta_{X_i|j}\lambda_j}{\eta'_{X_i|t}\lambda'_t / \sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j})] \leq \log[E_q(\frac{\eta_{X_i|t}\lambda_t / \sum_{j=1}^{k} \eta_{X_i|j}\lambda_j}{\eta'_{X_i|t}\lambda'_t / \sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j})] \\
\Rightarrow &\sum_{t=1}^{k} \log(\frac{\eta_{X_i|t}\lambda_t / \sum_{j=1}^{k} \eta_{X_i|j}\lambda_j}{\eta'_{X_i|t}\lambda'_t / \sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j})\frac{\eta'_{X_i|t}\lambda'_t}{\sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j} \leq \log \sum_{t=1}^{k} \eta_{X_i|t}\lambda_t / \sum_{j=1}^{k} \eta_{X_i|j}\lambda_j = 0 \\
\Rightarrow &\sum_{t=1}^{k} \log(\frac{\eta_{X_i|t}\lambda_t}{\eta'_{X_i|t}\lambda'_t})\frac{\eta'_{X_i|t}\lambda'_t}{\sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j} + \log(\sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j) \leq \log(\sum_{j=1}^{k} \eta_{X_i|j}\lambda_j)
\end{aligned}$$

Thus, we have the lower bound of $\log(\sum_{j=1}^{k} \eta_{X_i|j}\lambda_j)$ and then the lower bound of the $\ell(\theta|D_1 \cup D_2)$ as

$$\sum_{i=1}^{n}(\log \eta_{X_i|Y_i} + \log \lambda_{Y_i}) + \sum_{i=1}^{n+m}\{\sum_{t=1}^{k} \log(\frac{\eta_{X_i|t}\lambda_t}{\eta'_{X_i|t}\lambda'_t})\frac{\eta'_{X_i|t}\lambda'_t}{\sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j} + \log(\sum_{j=1}^{k} \eta'_{X_i|j}\lambda'_j)\}.$$

Based on the lower bound, the EM algorithm maximize the lower bound in the M step.

5. The PMF is

$$P[X = n] = \binom{n-1}{2} p^3(1-p)^{n-3}$$

for $n \geq 3$.

(a) Look at the likelihood ratio as

$$\frac{L(p)}{L(1/2)} = \frac{p^3(1-p)^{n-3}}{(1/2)^n}$$

which is increasing in n for $p < 1/2$. Thus, the UMP test rejects H_0 if X is large. The test function is

$$\phi(x) = \begin{cases} 0 & \text{when } X < n_0 \\ \gamma & \text{when } X = n_0 \\ 1 & \text{when } X > n_0 \end{cases}$$

where n_0 and γ are determined by

$$\sum_{n=n_0+1}^{\infty} (\frac{n-1}{2})(1/2)^n < \alpha,$$
$$\sum_{n=n_0}^{\infty} (\frac{n-1}{2})(1/2)^n \geq \alpha,$$
$$\gamma = \frac{\alpha - \sum_{n=n_0+1}^{\infty}(\frac{n-1}{2})(1/2)^n}{(\frac{n_0-1}{2})(1/2)^{n_0}}.$$

(b) When $n = 12$ is observed,

$$\sum_{n=3}^{11} \binom{n-1}{2}(1/2)^n = 0.9673.$$

Thus we reject H_0 at $\alpha = 0.05$.

6. (a) Since $\bar{X}$ is the CSS and it is an unbiased estimator of λ, it is UMVUE and so it is the best unbiased estimator of λ.

(b) Since $E(S^2) = \lambda$ and $\bar{X}$ is CSS, $E(S^2|barX)$ is the UMVUE of λ. Since UMVUE is unique, $E(S^2|\bar{X}) = \bar{X}$. The second part can be proven as

$$V(S^2) = E[V(S^2|\bar{X})] + V[E(S^2|\bar{X})] = E[V(S^2|\bar{X})] + V(\bar{X}) > V(\bar{X}).$$

(c) Sure. The conclusion holds for any unbiased estimator of λ.