DEPARTMENT OF STATISTICS
PHD QUALIFYING EXAM
STATISTICAL THEORY: Friday, Jan 7, 2005, 8:00-12:00 am

1. (a) $\hat{\theta}_2 = \max_{ij} X_{ij}$ and $\hat{\theta}_{1j} = X_{i(1)} = \min_j X_{ij}$.

   (b)
$$\tau(\theta) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2}\frac{\theta_2 + \theta_{i1}}{2} = \theta_2 + \frac{1}{n}\sum_{i=1}^{n}\theta_{i1}$$
and
$$\tau(\hat{\theta}) = X_{(2n)} + \frac{1}{n}\sum_{i=1}^{n} X_{i(1)}$$
It is not hard to show that $X_{(2n)} \xrightarrow{p} \theta_2$ and
$$\frac{1}{n}\sum_{1}^{n} X_{i(1)} - \frac{1}{n}\sum_{1}^{n} E_\theta(X_{i(1)}) \xrightarrow{p} 0$$
(since $\mathrm{Var}(\frac{1}{n}\sum_1^n X_{i(1)}) \to 0$). However,
$$\frac{1}{n}\sum_{1}^{n} E_\theta(X_{i(1)}) - \frac{1}{n}\sum_{1}^{n}\theta_{i1}$$
does not converge to zero so $\tau(\hat{\theta}) - \tau(\theta)$ does not converge to zero in probability.

2. Let $\theta = (\beta_1, \beta_2, \sigma_0^2)$. The likelihood function is
$$L(\theta) = (\frac{1}{\sqrt{2\pi}\sigma_0})^n(\prod_{i=1}^{n} w_i)e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} w_i[Y_i - \beta_1 - \beta_2(z_i - \bar{z})]^2}$$
and the loglikelihood function is
$$l(\theta) = -n\log(\sqrt{2\pi}) + \sum_{i=1}^{n}\log(w_i) - \frac{n}{2}\log(\sigma_0^2) - \frac{1}{2\sigma_0^2}\sum_{i=1}^{n} w_i[Y_i - \beta_1 - \beta_2(z_i - \bar{z})]^2.$$
Then, we have
$$\begin{aligned}
\frac{\partial \ell}{\partial \beta_1} =& \frac{1}{\sigma_0^2}\sum_{i=1}^{n} w_i[Y_i - \beta_1 - \beta_2(z_i - \bar{z})] = \frac{1}{\sigma_0^2}\sum_{i=1}^{n} w_i(Y_i - \beta_1)\\
\frac{\partial \ell}{\partial \beta_2} =& \frac{1}{\sigma_0^2}\sum_{i=1}^{n} w_i[Y_i - \beta_1 - \beta_2(z_i - \bar{z})](z_i - \bar{z}) = \frac{1}{\sigma_0^2}\sum_{i=1}^{n} w_i[Y_i - \beta_2(z_i - \bar{z})](z_i - \bar{z})\\
\frac{\partial \ell}{\partial \sigma_0^2} =& -\frac{n}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum_{i=1}^{n} w_i[Y_i - \beta_1 - \beta_2(z_i - \bar{z})]^2.
\end{aligned}$$

(a) Let

$$\frac{\partial \ell}{\partial \beta_1} = \frac{\partial \ell}{\partial \beta_2} = 0.$$

We have the MLE

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n w_i Y_i}{\sum_{i=1}^n w_i}$$

and

$$\hat{\beta}_2 = \frac{\sum_{i=1}^n w_i (z_i - \bar{z}) Y_i}{\sum_{i=1}^n w_i (z_i - \bar{z})^2}.$$

Straightforwardly, we have $E(\hat{\beta}_1) = \beta_1$, $E(\hat{\beta}_2) = \beta_2$,

$$V(\hat{\beta}_1) = \frac{\sigma_0^2}{\sum_{i=1}^n w_i}$$

and

$$V(\hat{\beta}_2) = \frac{\sigma_0^2}{\sum_{i=1}^n w_i (z_i - \bar{z})^2}.$$

(b) We still have $E(\hat{\beta}_1(\hat{w})) = \beta_1$ and $E(\hat{\beta}_2(\hat{w})) = \beta_2$. The proof is easy. Suppose $\hat{w}$ is estimated by $Z_1, \cdots, Z_m$. Then, we have

$$E(\hat{\beta}_1(\hat{w})|Z_1, \cdots, Z_m) = E(\beta_1|Z_1, \cdots, Z_m) = \beta_1.$$

(c) Since

$$\begin{aligned}
V(\hat{\beta}_1(\hat{w})) =& E[V(\hat{\beta}_1(\hat{w})|Z_1, \cdots, Z_m)] + V[E(\hat{\beta}_1(\hat{w}))|Z_1, \cdots, Z_m] \\
=& \sigma_0^2 E(\frac{1}{\sum_{i=1}^n \hat{w}_i}|Z_1, \cdots, Z_m) \\
\geq& \frac{\sigma_0^2}{E(\sum_{i=1}^n \hat{w}_i|Z_1, \cdots, Z_m)} \\
=& V(\hat{\beta}_1).
\end{aligned}$$

by Jessen inequality.

3. (a) Let X' be the true number. Then $P[X' = k] = \frac{\lambda^k}{k!} e^{-\lambda}$, $P[X = 0|X' = 1] = \theta$, $P[X = 1|X' = 1] = 1 - \theta$ and $P[X = X'|X' \neq 1] = 1$. Let $p(k) = P[X = k]$. Then, we have

$$p(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda}, & \text{when } k \geq 2, \\ (1-\theta)\lambda e^{-\lambda}, & \text{when } k = 1, \\ (1+\theta\lambda) e^{-\lambda}. & \end{cases}$$

We can also write it as

$$\begin{aligned}
p(k) &= \left(\frac{\lambda^k}{k!} e^{-\lambda} \right)^{I_{k\geq 2}} \left((1-\theta)\lambda e^{-\lambda} \right)^{I_{k=1}} \left((1+\theta\lambda) e^{-\lambda} \right)^{I_{k=0}} \\
&= \left(\frac{\lambda^k}{k!} e^{-\lambda} \right) (1-\theta)^{I_{k=1}} (1+\theta\lambda)^{I_{k=0}}.
\end{aligned}$$

(b) The loglikelihhod function is

$$\ell(\lambda,\theta) = -\sum_{i=1}^{n} \log k! + \sum_{i=1}^{n} x_i \log(\lambda) - n\lambda + \sum_{i=1}^{n} I_{x_i=1} \log[(1-\theta)] + \sum_{i=1}^{n} I_{x_i=0} \log(1+\theta\lambda).$$

Thun, we have

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda} &= \frac{\sum_{i=1}^{n} X_i}{\lambda} - n + \frac{\theta}{1+\theta\lambda}\sum_{i=1}^{n} I_{X_i=0}\\ \frac{\partial \ell}{\partial \theta} &= -\frac{1}{1-\theta}\sum_{i=1}^{n} I_{X_i=1} + \frac{\lambda}{1+\theta\lambda}\sum_{i=1}^{n} I_{X_i=0}\end{aligned}$$

Let $r_1 = \sum_{i=1}^{n} I_{X_i=1}/n$ and $r_0 = \sum_{i=1}^{n} I_{X_i=1}/n$. Then, when $r_1 > 0$, we have

$$\begin{aligned}\hat{\theta} =& \frac{1}{2\bar{X}(r_1+r_0)}[(r_1^2 - r_1 + r_1\bar{X} + 2\bar{X}r_0 + r_1 r_0)\\ &- \sqrt{(r_1^2 - r_1 + r_1\bar{x} + 2\bar{X}r_0 + r_1 r_0)^2 - 4\bar{X}(r_1+r_0)(\bar{X}r_0 + r_1 r_0 + r_1)}]\end{aligned}$$

and

$$\hat{\lambda} = \bar{X} + \frac{r_1}{1-\hat{\theta}}.$$

When $r_1 = 0$, we have $\hat{\theta} = 0$ and $\hat{\lambda} = \bar{X}$.

4. (a) Suppose $T = f(X)$ is an unbiased estimator. Then, we have

$$E[f(X)] = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma^2$$

for all $\mu \in \infty$, which implies that

$$\int_{-\infty}^{\infty} \frac{f(x) e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} e^{-\frac{\mu}{\sigma}x} dx = \sigma^2 e^{\frac{\mu^2}{2\sigma^2}}$$

for all μ. By taking μ is 0, we have

$$\int_{0}^{\infty} \frac{f(x)+f(-x)}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \sigma^2.$$

Recall the uniqueness of the Lapalace Transformation, we have the uniqueness of $f(x) + f(-x)$. Note that $f(x) + f(-x) = x^2$ satisfies the condition. We have

$$f(x) + f(-x) = x^2.$$

Note that by replacing μ by $-\mu$ in the first equation, we have

$$\int_{-\infty}^{\infty} \frac{f(-x)}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma^2$$

which implies that

$$\int_{-\infty}^{\infty} \frac{\frac{f(x)+f(-x)}{2}}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma^2.$$

However, this is only true when $\mu = 0$. Thus, there is no such $f(X)$. Thus, there is no unbiased estimator for σ^2.

(b) Note that $E(X^2) = \mu^2 + \sigma^2 = \mu^2 + 1$ and

$$\begin{aligned} E(X^4) = &\int_{-\infty}^{\infty} \frac{x^4}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4 \\ =&3 + 6\mu^2 + \mu^4. \end{aligned}$$

Let

$$T = X^4 - 6(X^2 - 1) - 3.$$

Then, we have $E(T) = \mu^4$.

(c) Suppose $T = f(X)$ is an unbiased estimator of $|\mu|$. Then, we have

$$E(T) = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = |\mu|.$$

We can do the similar inference as we did in (a).

5. (a)

$$\begin{aligned} E(X) =&\frac{1-\theta^2}{2}[\int_{-\infty}^{0} x e^{\theta x + x} dx + \int_{0}^{\infty} x e^{\theta x - x} dx] \\ =&\frac{1-\theta^2}{2}[-\frac{1}{(\theta+1)^2} + \frac{1}{(\theta-1)^2}] \\ =&\frac{2\theta}{(\theta-1)(\theta+1)}. \end{aligned}$$

and

$$\begin{aligned} E(X^2) =&\frac{1-\theta^2}{2}[\int_{-\infty}^{0} x^2 e^{\theta x + x} dx + \int_{0}^{\infty} x^2 e^{\theta x - x} dx] \\ =&\frac{1-\theta^2}{2}[\frac{2}{(\theta+1)^3} - \frac{2}{(\theta-1)^3}] \\ =&\frac{6\theta^2 + 2}{(\theta-1)^2(\theta+1)^2}. \end{aligned}$$

Thus, we have

$$V(X) = \frac{2(\theta^2+1)}{(\theta-1)^2(\theta+1)^2}.$$

Thus, we have

$$\begin{aligned} E_\theta[aX - E_\theta(X)] =&a^2 V_\theta(X) + (a-1)^2 E_\theta^2(X) \\ =&\frac{2a^2(\theta^2+1) + 4\theta^2(a-1)^2}{(\theta-1)^2(\theta+1)^2} \end{aligned}$$

(b) Consider the inequality

$$E_\theta[aX - E_\theta(X)]^2 \leq V_\theta(X).$$

That is

$$2a^2(\theta^2+1) + 4\theta^2(a-1)^2 \leq 2(\theta^2+1) \Leftrightarrow \theta^2 \leq \frac{1-a^2}{3a^2-4a+1}.$$

Whan $a \in (0,1)$, the above is always less than 0. Thus, we have

$$E_\theta[aX - E_\theta(X)]^2 < V_\theta(X)$$

for all $0 < a < 1$. Thus, X is inadmissible.

6. Let p be the probablity that A or B find an error. Then set up a 2 by 2 contingency table with p and $1-p$ as the probabilities in the margins and 1, 2, 3 and $N-6$ in the table.

	p	$1-p$
p	1	2
$1-p$	3	$N-6$

Using independence the likelihood function is then

$$L = \frac{N!}{1!2!3!(N-6)!}p^7(1-p)^{2N-7}$$

If we differentiate on p we get $2Np = 7$. You have to be carefull with the N since the answer is on the 'boundary' at $N = 6$.