1. (a) $\hat{\theta}_2 = \max_{i,j} X_{ij}$ and $\hat{\theta}_1 = X_{i(1)} = \min_j X_{ij}$.

(b) 

$$\tau(\theta) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{2} \frac{\theta_2 + \theta_{i1}}{2} = \theta_2 + \frac{1}{n} \sum_{i=1}^{n} \theta_{i1}$$

and

$$\tau(\hat{\theta}) = X_{(2n)} + \frac{1}{n} \sum_{i=1}^{n} X_{i(1)}$$

It is not hard to show that $X_{(2n)} \overset{p}{\rightarrow} \theta_2$ and

$$\frac{1}{n} \sum_{i=1}^{n} X_{i(1)} - \frac{1}{n} \sum_{i=1}^{n} E_\theta(X_{i(1)}) \overset{p}{\rightarrow} 0$$

(since $\text{Var}(\frac{1}{n} \sum_{i=1}^{n} X_{i(1)}) \rightarrow 0$). However,

$$\frac{1}{n} \sum_{i=1}^{n} E_\theta(X_{i(1)}) - \frac{1}{n} \sum_{i=1}^{n} \theta_{i1}$$

does not converge to zero so $\tau(\hat{\theta}) - \tau(\theta)$ does not converge to zero in probability.

2. Let $\theta = (\beta_1, \beta_2, \sigma_0^2)$. The likelihood function is

$$L(\theta) = \left(\frac{1}{\sqrt{2\pi} \sigma_0}\right)^n \left(\prod_{i=1}^{n} w_i\right) e^{-\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} w_i [Y_i - \beta_1 - \beta_2 (z_i - \bar{z})]^2}$$

and the loglikelihood function is

$$l(\theta) = -n \log(\sqrt{2\pi}) + \sum_{i=1}^{n} \log(w_i) - \frac{n}{2} \log(\sigma_0^2) - \frac{1}{2\sigma_0^2} \sum_{i=1}^{n} w_i [Y_i - \beta_1 - \beta_2 (z_i - \bar{z})]^2.$$
(a) Let
\[ \frac{\partial \ell}{\partial \beta_1} = \frac{\partial \ell}{\partial \beta_2} = 0. \]
We have the MLE
\[ \hat{\beta}_1 = \frac{\sum_{i=1}^{n} w_i Y_i}{\sum_{i=1}^{n} w_i} \]
and
\[ \hat{\beta}_2 = \frac{\sum_{i=1}^{n} w_i (z_i - \hat{z}) Y_i}{\sum_{i=1}^{n} w_i (z_i - \hat{z})^2}. \]
Straightforwardly, we have \( E(\hat{\beta}_1) = \beta_1 \), \( E(\hat{\beta}_2) = \beta_2 \),
\[ V(\hat{\beta}_1) = \frac{\sigma_0^2}{\sum_{i=1}^{n} w_i} \]
and
\[ V(\hat{\beta}_2) = \frac{\sigma_0^2}{\sum_{i=1}^{n} w_i (z_i - \hat{z})^2}. \]
(b) We still have \( E(\hat{\beta}_1(\hat{w})) = \beta_1 \) and \( E(\hat{\beta}_2(\hat{w})) = \beta_2 \). The proof is easy. Suppose \( \hat{w} \) is estimated by \( Z_1, \ldots, Z_m \). Then, we have
\[ E(\hat{\beta}_1(\hat{w})|Z_1, \ldots, Z_m) = E(\beta_1|Z_1, \ldots, Z_m) = \beta_1. \]
(c) Since
\[ V(\hat{\beta}_1(\hat{w})) = E[V(\hat{\beta}_1(\hat{w})|Z_1, \ldots, Z_m)] + V[E(\hat{\beta}_1(\hat{w})|Z_1, \ldots, Z_m)] \]
\[ = \sigma_0^2 E(\frac{1}{\sum_{i=1}^{n} \hat{w}_i}|Z_1, \ldots, Z_m) \]
\[ \geq \frac{\sigma_0^2}{E(\sum_{i=1}^{n} \hat{w}_i|Z_1, \ldots, Z_m)} \]
\[ = V(\hat{\beta}_1). \]
by Jessen inequality.

3. (a) Let \( X' \) be the true number. Then \( P[X' = k] = \frac{\lambda^k}{k!} e^{-\lambda}, P[X = 0|X' = 1] = \theta, P[X = 1|X' = 1] = 1 - \theta \) and \( P[X = X'|X' \neq 1] = 1 \). Let \( p(k) = P[X = k] \). Then, we have
\[ p(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda}, & \text{when } k \geq 2, \\ (1 - \theta)\lambda e^{-\lambda}, & \text{when } k = 1, \\ (1 + \theta\lambda)e^{-\lambda}. & \end{cases} \]
We can also write it as
\[ p(k) = \left( \frac{\lambda^k}{k!} e^{-\lambda} \right)^{I_{k \geq 2}} \left( (1 - \theta)\lambda e^{-\lambda} \right)^{I_{k = 1}} \left( (1 + \theta\lambda)e^{-\lambda} \right)^{I_{k = 0}} \]
\[ = \left( \frac{\lambda^k}{k!} e^{-\lambda} \right)^{I_{k \geq 2}} (1 - \theta)^{I_{k = 1}} (1 + \theta\lambda)^{I_{k = 0}}. \]
(b) The loglikelihood function is
\[ \ell(\lambda, \theta) = - \sum_{i=1}^{n} \log k! + \sum_{i=1}^{n} \chi_i \log(\lambda) - n \lambda + \sum_{i=1}^{n} I_{X_i=1} \log[(1 - \theta)] + \sum_{i=1}^{n} I_{X_i=0} \log(1 + \theta \lambda). \]

Then, we have
\[ \frac{\partial \ell}{\partial \lambda} = \frac{\sum_{i=1}^{n} \chi_i}{\lambda} - n + \frac{\theta}{1 + \theta \lambda} \sum_{i=1}^{n} I_{X_i=0} \]
\[ \frac{\partial \ell}{\partial \theta} = - \frac{1}{1 - \theta} \sum_{i=1}^{n} I_{X_i=1} + \frac{\lambda}{1 + \theta \lambda} \sum_{i=1}^{n} I_{X_i=0} \]

Let \( r_1 = \sum_{i=1}^{n} I_{X_i=1}/n \) and \( r_0 = \sum_{i=1}^{n} I_{X_i=0}/n \). Then, when \( r_1 > 0 \), we have
\[ \hat{\theta} = \frac{1}{2\bar{X}(r_1 + r_0)}[(r_1^2 - r_1 + r_1 \bar{X} + 2\bar{X} r_0 + r_1 r_0)
- \sqrt{(r_1^2 - r_1 + r_1 \bar{X} + 2\bar{X} r_0 + r_1 r_0)^2 - 4\bar{X}(r_1 + r_0)(\bar{X} r_0 + r_1 r_0 + r_1)}] \]
and
\[ \hat{\lambda} = \bar{X} + \frac{r_1}{1 - \hat{\theta}}. \]

When \( r_1 = 0 \), we have \( \hat{\theta} = 0 \) and \( \hat{\lambda} = \bar{X} \).

4. (a) Suppose \( T = f(X) \) is an unbiased estimator. Then, we have
\[ E[f(X)] = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma^2 \]
for all \( \mu \in \mathbb{R} \), which implies that
\[ \int_{-\infty}^{\infty} \frac{f(x)e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma}} e^{-\frac{y^2}{2\sigma^2}} dx = \sigma^2 e^{\frac{\mu^2}{2\sigma^2}} \]
for all \( \mu \). By taking \( \mu = 0 \), we have
\[ \int_{0}^{\infty} \frac{f(x) + f(-x)}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} dx = \sigma^2. \]

Recall the uniqueness of the Laplace Transformation, we have the uniqueness of \( f(x) + f(-x) \). Note that \( f(x) + f(-x) = x^2 \) satisfies the condition. We have
\[ f(x) + f(-x) = x^2. \]

Note that by replacing \( \mu \) by \(-\mu\) in the first equation, we have
\[ \int_{-\infty}^{0} \frac{f(-x)}{\sqrt{2\pi\sigma}} e^{-\frac{(x+\mu)^2}{2\sigma^2}} dx = \sigma^2 \]
which implies that

$$\int_{-\infty}^{\infty} \frac{f(x) + f(-x)}{2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx = \sigma^2.$$  

However, this is only true when \( \mu = 0 \). Thus, there is no such \( f(X) \). Thus, there is no unbiased estimator for \( \sigma^2 \).

(b) Note that \( E(X^2) = \mu^2 + \sigma^2 = \mu^2 + 1 \) and

$$E(X^4) = \int_{-\infty}^{\infty} \frac{x^4}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$$
$$= 3 + 6\mu^2 + \mu^4.$$

Let

$$T = X^4 - 6(X^2 - 1) - 3.$$  

Then, we have \( E(T) = \mu^4 \).

(c) Suppose \( T = f(X) \) is an unbiased estimator of \( |\mu| \). Then, we have

$$E(T) = \int_{-\infty}^{\infty} \frac{f(x)}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx = |\mu|.$$  

We can do the similar inference as we did in (a).

5. (a)

$$E(X) = \frac{1 - \theta^2}{2} \left[ \int_{-\infty}^{0} xe^{\theta x^2} \, dx + \int_{0}^{\infty} xe^{\theta x^2} \, dx \right]$$
$$= \frac{1 - \theta^2}{2} \left[ \frac{1}{(\theta + 1)^2} + \frac{1}{(\theta - 1)^2} \right]$$
$$= \frac{2\theta}{(\theta - 1)(\theta + 1)}.$$

and

$$E(X^2) = \frac{1 - \theta^2}{2} \left[ \int_{-\infty}^{0} x^2 e^{\theta x^2} \, dx + \int_{0}^{\infty} x^2 e^{\theta x^2} \, dx \right]$$
$$= \frac{1 - \theta^2}{2} \left[ \frac{2}{(\theta + 1)^3} - \frac{2}{(\theta - 1)^3} \right]$$
$$= \frac{6\theta^2 + 2}{(\theta - 1)^2(\theta + 1)^2}.$$

Thus, we have

$$V(X) = \frac{2(\theta^2 + 1)}{(\theta - 1)^2(\theta + 1)^2}.$$  

Thus, we have

$$E_{\theta} [aX - E_{\theta}(X)] = a^2 V_{\theta}(X) + (a - 1)^2 E_{\theta}^2(X)$$
$$= \frac{2a^2(\theta^2 + 1) + 4\theta^2(a - 1)^2}{(\theta - 1)^2(\theta + 1)^2}.$$
(b) Consider the inequality

\[ E_\theta[aX - E_\theta(X)]^2 \leq V_\theta(X). \]

That is

\[ 2a^2(\theta^2 + 1) + 4\theta^2(a - 1)^2 \leq 2(\theta^2 + 1) \iff \theta^2 \leq \frac{1 - a^2}{3a^2 - 4a + 1}. \]

When \( a \in (0, 1) \), the above is always less than 0. Thus, we have

\[ E_\theta[aX - E_\theta(X)]^2 < V_\theta(X) \]

for all \( 0 < a < 1 \). Thus, \( X \) is inadmissible.

6. Let \( p \) be the probability that A or B find an error. Then set up a 2 by 2 contingency table with \( p \) and \( 1 - p \) as the probabilities in the margins and 1, 2, 3 and \( N - 6 \) in the table.

\[
\begin{array}{ccc}
  & p & 1 - p \\
 p & 1 & 2 \\
 1 - p & 3 & N - 6 \\
\end{array}
\]

Using independence the likelihood function is then

\[ L = \frac{N!}{1!2!3!(N - 6)!} p^7 (1 - p)^{2N - 7} \]

If we differentiate on \( p \) we get \( 2Np = 7 \). You have to be careful with the \( N \) since the answer is on the 'boundary' at \( N = 6 \).