Solutions to Theory in Winter 2004

1. (a) The density is

$$f(x|\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$$

and $E(X) = \theta$. The density of $nX_{(1)}$ can be computed straightforwardly. Note that $F(x) = 1 - e^{-x/\theta}$. We have

$$P(nX_{(1)} > x) = \prod_{i=1}^{n} P(X_i > \frac{x}{n}) = (e^{-\frac{x}{n\theta}})^n = e^{-\frac{x}{\theta}}$$

and so the density of $nX_{(1)}$ is still the same as $f(x|\theta)$ above. Thus, $E(nX_{(1)}) = \theta$.

(b) Clearly $V(nX_{(1)}) = \theta^2$. The loglikelihood function is

$$l(\theta) = -n\log(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} X_i.$$

Then, we have

$$\ell''(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} X_i \Rightarrow I(\theta) = \frac{n}{\theta^2}.$$

The Cramer-Rao lower bound is $I^{-1}(\theta) = \theta^2/n$. Thus, it does not attain it.

(c) It is clear that the estimate is inconsistent.

2. (a) Note that $\bar{X}$ and $\bar{Y}$ are sufficient statistics and $\bar{X} \sim N(\mu_x, 1/n)$, $\bar{Y} \sim N(\mu_y, 1/n)$. The likelihood ratio is

$$L(\mu_0, \mu_x', \mu_y') = \frac{e^{-\frac{(\bar{x}-\mu_0)^2+(\bar{y}-\mu_0)^2}{2n}}}{e^{-\frac{(x-\mu_x')^2+(y-\mu_y')^2}{2n}}} = e^{\frac{1}{n}[x(\mu_0-\mu_x')+y(\mu_0-\mu_y')]-\frac{1}{2n}[2\mu_0^2-\mu_x'^2-\mu_y'^2]}.$$

Then, H_0' is rejected if

$$x(\mu_0 - \mu_x') + y(\mu_0 - \mu_y') \leq C.$$

Thus, if and only if

$$\frac{\mu_x' + \mu_y'}{2} = \mu_0, \mu_x' < \mu_y'$$

the most powerful test rejects H_0' in the form of $\bar{Y} - \bar{X}$ greater than a constant.

(b) We may consider the null hypothesis as $H_0' : \mu_x = \mu_y = \frac{\mu_x+\mu_y}{2}$ versus $H_1' : \mu_x < \mu_y$. In this case, by (a), the most powerful test rejects H_0' by $\bar{Y} - \bar{X} > C$ for a constant C. Since the rejection region does not change with respect to μ_x abd μ_y, the is the UMP test.

3. Clearly that $\hat{p}_3 = \bar{X}$. The posterior density is

$$g(p|x_1, \cdots, x_n) = \frac{\Gamma(n+1)}{\Gamma(\sum_{i=1}^n X_i + \frac{1}{2})\Gamma(n - \sum_{i=1}^n X_i + \frac{1}{2})} p^{\sum_{i=1}^n X_i - \frac{1}{2}} (1-p)^{n - \sum_{i=1}^n X_i - \frac{1}{2}}$$

indicating that

$$\hat{p}_1 = \frac{\sum_{i=1}^n X_i + 1/2}{n+1}$$

and

$$\hat{p}_2 = \frac{\sum_{i=1}^n X_i - 1/2}{n-1}.$$

Thus, we have

$$\max(\hat{p}_1, \hat{p}_2, \hat{p}_3) - \min(\hat{p}_1, \hat{p}_2, \hat{p}_3)$$

goes to 0 in probability.

4. The W be the value of X and Y that is not closer to θ. Then, the joint density of (W, X) is

$$f(w, z|\theta) = \frac{1}{\pi} e^{-\frac{(w-\theta)^2 + (z-\theta)^2}{2}} = \frac{1}{\pi} e^{-[(\theta - \frac{w+z}{2})^2 + \frac{(x-w)^2}{4}]}, |z - \theta| \le |w - \theta|.$$

Under uniform prior for θ, the marginal density of (X, W) is

$$\begin{aligned} m(x, w) =& \int_{|z-\theta| \le |w-\theta|} f(w, z|\theta) d\theta \\ =& \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-w)^2}{4}}. \end{aligned}$$

The posterior density of θ is

$$g(\theta|w, z) = \sqrt{\frac{2}{\pi}} e^{-(\theta - \frac{z+w}{2})^2}, |z - \theta| \le |w - \theta|.$$

Thus, we have the posterior mean

$$\tilde{\theta} = E(\theta|z, w) = \begin{cases} \frac{z+w}{2} - \frac{1}{\sqrt{2\pi}}, & \text{when } z < w \\ \frac{z+w}{2} + \frac{1}{\sqrt{2\pi}}, & \text{when } w < z. \end{cases}$$

It is clear that $\tilde{\theta}$ is unbiased. After a few steps of computation, we have

$$\begin{aligned} V(\tilde{\theta}) =& E\{[(\frac{z+w}{2} - \theta)^2 - \frac{2}{\sqrt{2\pi}}(\frac{z+w}{2} - \theta) + \frac{1}{2\pi}] I_{z<w}\} \\ &+ E\{[(\frac{z+w}{2} - \theta)^2 + \frac{2}{\sqrt{2\pi}}(\frac{z+w}{2} - \theta) + \frac{}{2\pi}] I_{z>w}\} \\ =& \frac{1}{2} + \frac{1}{2\pi} - \frac{2}{\sqrt{2\pi}} E[(\frac{z+w}{2} - \theta) I_{z<w}] + \frac{2}{\sqrt{2\pi}} E[(\frac{z+w}{2} - \theta) I_{z>w}]. \end{aligned}$$

Now, consider

$$\begin{aligned}
E[(\frac{z+w}{2}-\theta)I_{z<w}] &= \int_\theta^\infty \int_{2\theta-w}^{w} (\frac{z+w}{2}-\theta)\frac{1}{\pi}e^{-\frac{(w-\theta)^2+(z-\theta)^2}{2}}dzdw \\
&= \int_\theta^\infty \frac{(w-\theta)}{\sqrt{2\pi}}[2\Phi(w-\theta)-1]e^{-\frac{(w-\theta)^2}{2}}dw \\
&= \int_\theta^\infty \frac{2(w-\theta)}{\sqrt{2\pi}}\Phi(w-\theta)e^{-\frac{(w-\theta)^2}{2}}dw - \frac{1}{\sqrt{2\pi}} \\
&= \frac{1}{2\sqrt{\pi}}.
\end{aligned}$$

Similarly, we can show

$$E[(\frac{z+w}{2}-\theta)I_{z>w}] = -\frac{1}{2\sqrt{\pi}}.$$

Thus, we have

$$V(\hat{\theta}) = \frac{1}{2} + \frac{1}{2\pi} - \frac{\sqrt{2}}{\pi} = 0.2090.$$

We need to compute the density of Z. First, for any $z-\theta<0$, we have

$$\begin{aligned}
P(Z \le z) &= 2[P(Y-\theta \le X-\theta \le z-\theta) + P(-(Y-\theta) \le X-\theta \le z-\theta)] \\
&= 2\Phi^2(z-\theta).
\end{aligned}$$

Similarly, we have for $z-\theta>0$,

$$P(Z>z) = 2\Phi^2(\theta-z) \Rightarrow P(Z<z) = 1-2\Phi^2(\theta-z).$$

Thus, the CDF of Z is

$$F_Z(z) = \begin{cases} 2\Phi^2(z-\theta), & \text{when } z \le \theta, \\ 1-2\Phi^2(\theta-z), & \text{when } z > \theta. \end{cases}$$

The density is

$$f_Z(z) = 4\Phi(-|z-\theta|)\phi(z-\theta)$$

and $E(Z)=\theta$, $V(Z) = 1-2/\pi$.

(a) The risk function is

$$E_\theta(Z-\theta)^2 = 1-\frac{2}{\pi} = 0.3633 > E_\theta(\tilde{\theta}-\theta)^2.$$

Thus, Z is inadmissible.

(b) The MES is $V(Z) = 1-2/\pi$.

5. (a) Given $N = n_1 + n_2 + n_3$, the probability mass function is

$$p(n_1, n_2, n_3) = \frac{N!}{n_1!n_2!n_3!}(f^2)^{n_1}[2f(1-f)]^{n_2}[(1-f)^2]^{n_3}.$$

Its logarithm is

$$\log[p(n_1, n_2, n_3)] = \log(\frac{N!2^{n_2}}{n_1!n_2!n_3!}) + (2n_1 + n_2)\log(f) + (2n_3 + n_2)\log(1-f).$$

Thus, we have the MLE

$$\hat{f} = \frac{2n_1 + n_2}{2(n_1 + n_2 + n_3)} = \frac{2n_1 + n_2}{2N}.$$

(b) By SLLN, we have $\frac{n_1}{N} \xrightarrow{P} f^2$ and $\frac{n_2}{N} \xrightarrow{P} 2f(1-f)$. Thus,

$$\hat{f} \xrightarrow{P} f^2 + f(1-f) = f.$$