Solutions to Theory in Fall 2004

1. (a) Without loss of generality, we assume $a_1 = 1$ and $a_0 = 0$. The test function is

$$\phi(x) = \begin{cases} 1, & \text{if } x \in R, \\ 0, & \text{if } x \notin R. \end{cases}$$

Thus, the test as a decision function is

$$\delta(x) = \begin{cases} 1, & \text{if } x \in R, \\ 0, & \text{if } x \notin R. \end{cases}$$

(b) Note that

$$\begin{aligned} E_\theta[l(\theta, \delta(X))] &= \begin{cases} P_\theta(X \in R), & \text{if } \theta \in \Theta_0 \\ P_\theta(X \notin R), & \text{if } \theta \in \Theta_1, \end{cases} \\ &= \begin{cases} P_\theta(\delta(X) = 1), & \text{if } \theta \in \Theta_0 \\ P_\theta(\delta(X) = 0), & \text{if } \theta \in \Theta_1, \end{cases} \end{aligned}$$

and

$$E_\theta[l(\theta', \delta(X))] = \begin{cases} E_\theta[l(\theta, \delta(X))], & \text{if } \theta, \theta' \in \Theta_0 \text{ or } \theta, \theta' \in \Theta_1 \\ 1 - E_\theta[l(\theta, \delta(X))], & \text{if } \theta \in \Theta_0, \theta' \in \Theta_1 \text{ or } \theta \notin \Theta_0, \theta' \in \Theta_1. \end{cases}$$

This implies that $E_\theta[\delta(X)] \leq 1/2$ if $\theta \in \Theta_1$ and $E_\theta[\delta(X) \geq 1/2]$ if $\theta \in \Theta_1$. Thus, we have

$$\beta(\theta', \phi) \geq \sup_{\theta \in \Theta_0} [\delta(X) = 1], \ \forall \theta' \in \Theta_1.$$

(c) Consider $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$ when $X \sim N(\theta, 1)$. We reject H_0 if $X > z_\alpha$. It is clear that (2) is satisfied since

$$\beta(\theta, \delta) = 1 - \Phi(z_\alpha - \theta)$$

is an increasing function of θ. However, if $\theta \leq 0$ and $\theta' > 0$, then

$$E_\theta[\ell(\theta, \delta(X))] = P_\theta(X \geq Z_\alpha) = 1 - \Phi(Z_\alpha - \theta)$$

and

$$E_\theta[\ell(\theta', \delta(X))] = P_\theta(X \leq Z_\alpha) = \Phi(Z_\alpha - \theta).$$

If $\alpha > 0.5$, then $E_\theta[\ell(\theta, \delta(X))] > E_\theta[\ell(\theta', \delta(X))]$.

2. (a) The density of x_i is

$$f_\theta(x) = \frac{\gamma}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} + \frac{1-\gamma}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}.$$

Thus, the likelihood function is

$$L(\theta) = \prod_{i=1}^n \left[\frac{\gamma}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_i-\mu_1)^2}{2\sigma_1^2}} + \frac{1-\gamma}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_i-\mu_2)^2}{2\sigma_2^2}}\right].$$

(b) When λ is close to 0, the first term can be ignored, which indicates that the estimate of μ_1 and σ_1^2 could be anything. Thus, an MLE does not exist.

(c) In this case, the conditional density of x_i given z_i is

$$f_\theta(x|z) = \left(\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\right)^{2-z_i} \left(\frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}\right)^{z_i-1}$$

and the marginal mass function of z is

$$p_\lambda(z) = \gamma^{2-z_i}(1-\gamma)^{z_i-1}.$$

Thus, the likelihood is

$$L(\theta) = \left[\prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\right)^{2-z_i}\right] \left[\prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}\right)^{z_i-1}\right] \left[\gamma^{2n-\sum_{i=1}^n z_i}\right] \left[(1-\gamma)^{\sum_{i=1}^n z_i - n}\right].$$

Thus, it gives

$$\begin{aligned}
\hat{\mu}_1 &= \frac{\sum_{i=1}^n (2-z_i)x_i}{\sum_{i=1}^n (2-z_i)}, \\
\hat{\mu}_2 &= \frac{\sum_{i=1}^n (z_i-1)x_i}{\sum_{i=1}^n (z_i-1)}, \\
\hat{\sigma}_1^2 &= \frac{\sum_{i=1}^n (x_i-\hat{\mu}_1)^2(2-z_i)}{n}, \\
\hat{\sigma}_2^2 &= \frac{\sum_{i=1}^n (x_i-\hat{\mu}_2)^2(z_i-1)}{n}, \\
\hat{\lambda} &= 2-\bar{z}.
\end{aligned}$$

3. The likelihood ratio is

$$R(x) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\frac{1}{\pi(1+x^2)}} = \frac{\pi}{\sqrt{2\pi}}(1+x^2)e^{-\frac{x^2}{2}}.$$

Since

$$\frac{dR(x)}{dx^2} = \sqrt{\frac{\pi}{2}} \frac{1-x^2}{2} e^{-\frac{x^2}{2}}$$

is decreasing in x^2 only when $x > 1$, only when $c > 1$ that the most powerful test would be of the form $|x| > c$. In this case, the type I error probability is

$$P_{H_0}(|X| > c) = 2[1-\Phi(c)] < 2[1-\Phi(1)].$$

Thus, only when $\alpha < 2[1-\Phi(1)]$ that the most powerful test would be of the form $|X| > c$.

4. (a) Suppose the prior is $Beta(\alpha, \beta)$, then the posterior density is $Beta(\alpha+x, 2+\beta-x)$ as

$$g(p|x) = \frac{\Gamma(2+\alpha+\beta)}{\Gamma(\alpha+x)\Gamma(2+\beta-x)} p^{\alpha+x-1}(1-p)^{2+\beta-x-1}.$$

Under the square error loss, the Bayesian estimator is

$$\delta(x) = E(p|x) = \frac{\alpha+x}{2+\alpha+\beta}.$$

The risk function is

$$\begin{aligned} E_p(\delta(x)-p)^2 =&(1-p)^2(\frac{\alpha}{2+\alpha+\beta}-p)^2 + 2p(1-p)[\frac{\alpha+1}{2+\alpha+\beta}-p]^2 \\ &+ p^2[\frac{\alpha+2}{2+\alpha+\beta}-p]^2 \\ =&\frac{1}{(2+\alpha+\beta)^2}\{2p(1-p)+[\alpha(1-p)-\beta p]^2\}. \end{aligned}$$

If the risk is constant, then the Bayesian estimator is minimax. To make this, we only need to choose $\alpha = \beta = 1/\sqrt{2}$. In this case,

$$E_p((\delta(x)-p)) = \frac{1}{2(2+\sqrt{2})^2}.$$

(b) If the Bayesian estimatio is $\delta(x)$ as in (1), then it is minimax. It is minimax if

$$\delta(x) = \begin{cases} \frac{\sqrt{2}}{2(2+\sqrt{2})}, & \text{when } x=0, \\ \frac{1}{2}, & \text{when } x=1, \\ \frac{4+\sqrt{2}}{2(2+\sqrt{2})}, & \text{when } x=2. \end{cases}$$

Let us consider the prior as $\pi(p_0) = y$, $\pi(1/2) = 1-2y$ and $\pi(1-p_0) = y$. Then, then Bayesian estimation is

$$\delta(x) = \frac{yp_0^{x+1}(1-p_0)^{2-x} + (1-2y)(1/2)^3 + y(1-p_0)^{x+1}p_0^{2-x}}{yp_0^x(1-p_0)^{2-x} + (1-2y)(1/2)^2 + y(1-p_0)^x p_0^{2-x}}.$$

Then, we have $\delta(1) = 1/2$,

$$\delta(0) = \frac{yp_0(1-p_0) + \frac{1-2y}{8}}{y[p_0^2 + (1-y_0)^2] + \frac{1-2y}{4}}.$$

It is only need to make $\delta(0) = \frac{\sqrt{2}}{2(2+\sqrt{2})}$, then $\delta(x)$ is minimax. Let $p_0 = 1/128$. We have

$$\frac{2048-3969y}{4096+7938} = \frac{\sqrt{2}}{2(2+\sqrt{2})} \Rightarrow y = \frac{2048\sqrt{2}}{11907} = 0.2432.$$

Thus, we can choose prior as

$$\pi(\frac{1}{128}) = \pi(\frac{127}{128}) = 0.2432, \pi(\frac{1}{2}) = 0.5135.$$

Thus, it is minimax.