Solutions to Theory in Fall 2003

1. (a) The likelihood function is
$$L(\theta) = \theta^n e^{-\theta \sum_{i=1}^n X_i}$$
and its logarithm is
$$\ell(\theta) = n\log(\theta) - \theta \sum_{i=1}^n X_i.$$
Thus, we have
$$\ell'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n X_i \Rightarrow \hat{\theta} = \frac{1}{\bar{X}}.$$

   (b) Note that
$$\ell''(\theta) = -\frac{n}{\theta^2} \Rightarrow I(\theta) = \frac{n}{\theta^2}.$$
Thus,
$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{L} N(0, \theta^2).$$

   (c) Note that $P(Z_i = 1) = 1 - e^{-\theta c}$. Thus, the loglikelihood function
$$\ell(\theta) = n\bar{Z}\log(\frac{1 - e^{-\theta c}}{e^{-\theta c}}) - nc\theta.$$
Then, we have
$$\ell'(\theta) = n\bar{Z}\frac{ce^{\theta c}}{e^{\theta c} - 1} - nc \Rightarrow \hat{\theta} = -\frac{\log(1 - \bar{Z})}{c}.$$

   (d) Still
$$\ell''(\theta) = -n\bar{Z}\frac{c^2 e^{\theta c}}{(e^{\theta c} - 1)^2} \Rightarrow I(\theta) = E[-\ell''(\theta)] = \frac{nc^2}{e^{\theta c} - 1}.$$
Thus, we have
$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{L} N(0, \frac{e^{\theta c} - 1}{c^2}).$$

   (e) We need to look at the ratio of the asymptotic variance as
$$R(\theta) = \frac{\theta^2}{\frac{1}{c^2}(e^{\theta c} - 1)} = \frac{c^2\theta^2}{e^{\theta c} - 1}.$$
It is greater than 1 for small θ and less than 1 for large θ.

2. (a) Note that $P(X_1 > k) = \Phi(\mu - k)$ which is increasing in μ. The UMPU test for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ accepts H_0 if $\mu_0 - z_{\alpha/2}/\sqrt{n} \le \bar{X} \le \mu_0 + z_{\alpha/2}/\sqrt{n}$. Then, the $100(1-\alpha)\%$ confidence interval for $\Phi(\mu - k)$ is the region composed of μ_0 such that the observed $\bar{X}$ accept H_0, which is
$$\{p : p = \Phi(\mu_0 - k), \mu_0 \in [\bar{X} - z_{\alpha/2}/\sqrt{n}, \bar{X} + z_{\alpha/2}/\sqrt{n}]\}$$
$$=[\Phi(\bar{X} - \frac{z_{\alpha/2}}{\sqrt{n}} - k), \Phi(\bar{X} + \frac{z_{\alpha}/2}{\sqrt{n}} - k)]$$

(b) The length of the confidence interval is

$$L = \Phi(\bar{X} + \frac{z_{\alpha/2}}{\sqrt{n}} - k) - \Phi(\bar{X} - \frac{z_\alpha/2}{\sqrt{n}} - k).$$

Note that $\bar{X} \sim N(\mu, \frac{1}{n})$. We have

$$\begin{aligned}
E(L) =& \int_{-\infty}^{\infty} [\Phi(x + \frac{z_{\alpha/2}}{\sqrt{n}} - k) - \Phi(x - \frac{z_\alpha/2}{\sqrt{n}} - k)] \frac{1}{\sqrt{2\pi n}} e^{-\frac{(x-\mu)^2}{2n}} dx \\
=& \frac{1}{2\pi n} \left\{ \int_{-\infty}^{\infty} [\int_{x-\frac{z_{\alpha/2}}{\sqrt{n}}-k}^{x+\frac{z_{\alpha/2}}{\sqrt{n}}-k} e^{-\frac{y^2}{2}} dy] e^{-\frac{(x-\mu)^2}{2n}} dx \right\} \\
=& \frac{1}{2\pi n} \left\{ \int_{-\infty}^{\infty} [\int_{x-\frac{z_{\alpha/2}}{\sqrt{n}}-(k-\mu)}^{x+\frac{z_{\alpha/2}}{\sqrt{n}}-(k-\mu)} e^{-\frac{y^2}{2}} dy] e^{-\frac{x^2}{2n}} dx \right\} \\
=& \frac{1}{2\pi n}
\end{aligned}$$

5. The density is

$$f(x|\theta) = \frac{1}{\theta} I_{[\theta, 2\theta]}(X_i).$$

Let $X_{(1)} = \min(X_1, \cdots, X_n)$ and $X_{(n)} = \max(X_1, \cdots, X_n)$.

(a) The likelihood function is

$$L(\theta) = \frac{1}{\theta^n} I_{[\theta,2\theta]}(X_{(1)}) I_{[\theta,2\theta]}(X_{(n)}).$$

Note that we always have $X_{(1)} \geq X_{(n)}/2$. Thus the MLE is

$$\hat{\theta} = \frac{X_{(n)}}{2}.$$

By the definition of W_n, we have

$$W_{(n)} = X_{(1)}.$$

(b) It is clear that the denisty of $\hat{\theta}$ is

$$g_{\hat{\theta}}(x) = n \frac{(x-\theta)^{n-1}}{\theta^n}.$$

Thus, we have

$$E[(\hat{\theta}_n - \theta)^2] = \frac{\theta^2}{2(n+2)(n+1)}.$$

The joint density of $\hat{\theta}$ and W_n is

$$g_{\hat{\theta}, W_n}(x, y) = \frac{n(n-1)(x-y)^{n-2}}{\theta^n}, \theta < y < x < 2\theta.$$

Thus, we have

$$E[(W_n - \theta)^2] = \frac{2\theta^2}{(n+2)(n+1)}$$

and

$$E[(\hat{\theta}_n - \theta)(W_n - \theta)] = -\frac{\theta^2}{2(n+2)(n+1)}.$$

Thus, we have

$$E_\theta[(T_n - \theta)^2] = \frac{6\theta^2}{25(n+2)(n+1)}.$$

Thus,

$$\lim_{n\to\infty} \frac{E_\theta[(\hat{\theta}_n^2 - \theta)^2]}{E_\theta[(T_n - \theta)^2]} = \frac{25}{12}.$$

(c) Yes, it is a contradiction, but the asymptotic efficiency requires the support does not change.

6. Here $P(X_i = x_i) = 1/(N+1)$ for $x_i = 0, 1, \cdots, N$. Thus, the joint PMF is

$$F(x_1, x_2, x_3|N) = P(X_1 = x_1, X_2 = x_2, X_3 = x_3|N) = \frac{1}{(N+1)^3}.$$

The posterior PMF of N is

$$P[N = k|x_1, X_2, x_3] = \frac{\frac{1}{(k+1)^3}\frac{60}{137k}}{\sum_{k=1}^5 \frac{1}{(k+1)^3}\frac{60}{137k}} = \frac{[k(k+1)^3]^{-1}}{\sum_{k=1}^5 [k(k+1)^3]^{-1}}.$$

Then, we have the following posterior PMF table

1	2	3	4	5
0.8243	0.1221	0.0343	0.0132	0.0061

8. The CDF of $X_{(n)}$ is

$$G(x|\theta) = \frac{x^n}{\theta^n}$$

and the PDF is

$$g(x|\theta) = \frac{nx^n}{\theta^n}.$$

Thus, the P-value is given by

$$P = 1 - G(X_{(n)}|1) = \begin{cases} 1 - X_{(10)}^{10}, & \text{when } X_{(n)} \le 1 \\ 0, & \text{when } X_{(n)} > 1 \end{cases}.$$

(a) It is

$$E_1(P) = 1 - E_1(X_{(10)}^{10}) = \frac{1}{2}.$$

(b) Generally, we have

$$E_\theta(P) = E_\theta(1 - X_{(10)}^{10}) = \frac{1}{2\theta^{10}}.$$

When $\theta = 1.5$, it is 0.00867.