An Example of Inconsistent MLE of Spatial Covariance Parameters under Increasing Domain Asymptotics

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Abstract

Asymptotic properties of estimators of covariance parameters in spatial statistics are commonly considered under the frameworks of increasing domain and fixed domain asymptotics, respectively. Although inconsistency is a general conclusion under the framework of fixed domain asymptotics, it is generally believed that consistency should generally hold under the framework of increasing domain asymptotics. This article provides an example in which the maximum likelihood estimator (MLE) of covariance parameters is still inconsistent under the framework of increasing domain asymptotics. Therefore, consistency may still be a problem under the framework of increasing domain asymptotics.

Keywords: Fixed domain and increasing domain asymptotics, Gaussian random field, inconsistency, maximum likelihood estimator.

1. Introduction

Spatial data are often dependent, observations from nearby sites tending to be more likely to be close to each other than observations from distant sites. In practice, the usual method is to model the dependence by a parametric covariance function in a Gaussian random field, and then to use a likelihood-based approach to estimate the parameters of the covariance function. For the purposes of making inferences, asymptotic properties of the maximum likelihood estimator (MLE) of these parameters are important. However, the derivation of the asymptotic properties of the MLE is complicated by the fact that there are two different asymptotic frameworks: the fixed domain and increasing domain asymptotics (Cressie (1993), p. 350), where the fixed
domain asymptotics is also called the _infill asymptotics_. Under the framework of fixed domain asymptotics, the study region is fixed and bounded, and sampling locations are increasingly dense within the study region. Under the framework of increasing domain asymptotics, the study region increases without bound, and the minimum distance between sampled locations is bounded below by a positive constant.

It is not surprising that the asymptotic behavior of the MLE of covariance parameters in the spatial covariance function is quite different under the two frameworks, respectively. For example, it is known that the MLE is inconsistent under the framework of infill asymptotics (Du, Zhang, Mandrekar, 2009; Lahiri, 1996; Stein, 1999; Ying, 1991; Zhang, 2004). However, it is believed that these parameters should be consistently estimable under the framework of increasing domain asymptotics, since it has been shown in literature that with some regularity conditions their MLEs are consistent and asymptotically normal (Mardia and Marshall, 1984). Therefore, people believe that consistency should be the general conclusion under the framework of increasing domain asymptotics.

In this article, we provide an example in which the MLE is still inconsistent under the framework of increasing domain asymptotics. We derive the closed form expression of the likelihood function and show that the difference between the likelihood functions at distinct sets of parameters approaches a nondegenerate random variable in distribution, which can be larger than any positive value with a positive probability. Therefore, the consistency of the MLE does not hold. This example implies that consistency may still be a concern under the framework of increasing domain asymptotics.

2. The Example

The basic model in spatial statistics can be generally expressed as

$$Y(s) = \mu(s) + \delta(s) + \epsilon(s), s \in D \subseteq \mathbb{R}^d,$$


where $\mu(s)$ is the mean function, $\delta(s)$ is a spatially correlated process, and $\epsilon(s)$ is a white noise error process. The spatially correlated process $\delta(s)$ is commonly modeled by a stationary Gaussian process with $\mathbb{E}(\delta(s)) = 0$, $\mathbb{V}(\delta(s)) = \sigma^2$, and $\text{corr}(\delta(s), \delta(s+h)) = c(h)$. The error process $\epsilon(s)$ is often assumed iid $N(0, \tau^2)$. Under these two assumptions, there are $E(Y(s)) = \mu(s)$, $V(Y(s)) = \sigma^2 + \tau^2$, and $\text{Cov}(Y(s), Y(s+h)) = \sigma^2 c(h)$ if $h \neq 0$.  


In Model (1), the values of $\sigma^2$ and $\tau^2$ describe the strength of the spatial correlated effect and the nugget effect, respectively, where the nugget effect disappears if $\tau^2 = 0$.

Because the correlation function $c(h)$ must satisfy the positive-definiteness condition, it is often to specify a stationary parametric model as

$$c(h) = c(h; \theta) = \text{corr}(\delta(s), \delta(s + h)), \theta \in \Theta \in \mathbb{R}^q.$$  \hfill (2)

There are many parametric models proposed for $c(h; \theta)$ in literature. Among those, the most popular one is the Matérn correlation family (Matérn, 1986). The Matérn correlation family includes the exponential correlation function as a special case, where $q = 1$ and $c(h; \theta) = e^{-\theta \|h\|}, \theta > 0$.

In this article, we are interested in the construction of an example when the spatial correlation function is the exponential correlation function. In the example, we assume $\mu(s) = 0$, $\sigma^2 = 1$, and $\tau^2 = 0$ such that $E(Y(s)) = 0$, $V(Y(s)) = 1$, and $\text{corr}(Y(s), Y(s + h)) = e^{-\theta \|h\|}$ in Model (1). We attempt to show that the MLE of $\theta$ is inconsistent in the example. Since $\theta \in \mathbb{R}^+$ is the only parameter contained in our model, we use $\rho = e^{-\theta}$ to represent the unknown parameter instead. We assume $\rho \in [0, 1)$ in our model, where $\rho = 0$ is derived by letting $\theta \to \infty$. The problems of consistency are equivalent in the models using $\theta$ and $\rho$ as the unknown parameter, respectively.

Suppose spatial data are observed at irregularly spaced locations $s_1, \ldots, s_n$. Let $Y_i = Y(s_i)$ and $y = (Y_1, \ldots, Y_n)$. Then,

$$y \sim N(0, R),$$  \hfill (3)

where $R = (\rho \|s_i - s_j\|)_{n \times n}$ is the covariance (also the correlation) matrix of $y$. The loglikelihood function of $\rho$ is

$$\ell(\rho) = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \log |\det(R)| - \frac{1}{2} y^T R^{-1} y.$$  \hfill (4)

The MLE of $\rho$, which is denoted by $\hat{\rho}$, is derived by maximizing $\ell(\rho)$. The asymptotic property of $\hat{\rho}$ depends on the allocation of $S = \{s_1, \ldots, s_n\}$ as $n \to \infty$. It has been shown that $\hat{\rho}$ is consistent if $s_i = i$ under $d = 1$ (Mardia and Marshall, 1984). Here we consider another choice of $S$, where we set $s_i = 2i - 1$ for $i = 1, \ldots, n$. We show that $\hat{\rho}$ is an inconsistent estimator of $\rho$ as $n \to \infty$, which is enough to conclude that the MLE of $\theta$ is also an inconsistent estimator of $\theta$ in this case.
We focus on the proof of the inconsistency of \( \hat{\rho} \) under \( S = \{2^i - 1 : i = 1, \ldots, n\} \) in this article. The \((i, j)\)-th entry of \( R_\rho \) can be represented as
\[
r_{i,j}(\rho) = \rho^{2^{j-1} - 2^{i-1}}, \quad 1 \leq i \leq j \leq n,
\]
and \( r_{j,i}(\rho) = r_{i,j}(\rho) \) if \( i > j \). If \( n > 1 \), then the determinant of \( R_\rho \) is
\[
\det(R_\rho) = \prod_{i=1}^{n-1} (1 - \rho^{2^i}).
\]
Let \( V_\rho = R_\rho^{-1} \). Then, the \((i, j)\)-th entry of \( V_\rho \) can be expressed as
\[
\begin{align*}
v_{1,1}(\rho) &= \frac{1}{1 - \rho^2}, \\
v_{i,i}(\rho) &= \frac{\rho^{2^{i-1}}}{1 - \rho^{2^i}} + \frac{1}{1 - \rho^{2^n}}, \\
v_{n,n}(\rho) &= \frac{1}{1 - \rho^{2^{n-1}}}, \\
v_{i,i+1}(\rho) &= v_{i+1,i}(\rho) = -\frac{\rho^{2^{i-1}}}{1 - \rho^{2^n}},
\end{align*}
\]
and \( v_{i,j}(\rho) = 0 \) if \(|i - j| \geq 2\), where \( 1 < i < n \). If \( n > 1 \), then the loglikelihood function of \( \rho \) is
\[
\ell(\rho) = -\frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^{n-1} \log(1 - \rho^{2^i}) - \frac{1}{2} y^T V_\rho y
\]
\[
= -\frac{n}{2} \log(2\pi) - \frac{1}{2} \sum_{i=1}^{n-1} \log(1 - \rho^{2^i}) - \frac{1}{2} \sum_{i=1}^{n-1} \frac{(Y_i - \rho^{2^i} Y_{i+1})^2}{1 - \rho^{2^n}} + Y_n^2.
\]

**Lemma 1.** Let \( \rho_0 \) be the true parameter of \( \rho \) in Model (3) with \( s_i = 2^i - 1 \) for \( i = 1, \ldots, n \). Assume \( \rho_0 \in (0, 1) \). Define \( Z_i = (Y_i - \rho_0^{2^i-1} Y_{i+1})/\sqrt{1 - \rho_0^{2^i}} \) for \( i = 1, \ldots, n-1 \) and \( Z_n = Y_n \). Then, \( Z_1, \ldots, Z_n \) are iid \( N(0, 1) \).

**Proof:** It is clear that \( z = (Z_1, \ldots, Z_n) \) is a multivariate normal random vector with mean zero. Therefore, it is enough to show that the covariance
matrix of $z$ is the identity matrix. Straightforwardly, for any $1 \leq i < n$, we have

$$V(Z_i) = \frac{1}{1 - \rho_0^{2i}}[V(Y_i) - 2\rho_0^{2i-1}\text{Cov}(Y_i, Y_{i+1}) + \rho_0^{2i}V(Y_{i+1})] = 1$$

and

$$\text{Cov}(Z_i, Z_n) = \frac{\text{Cov}(Y_i, Y_n) - \rho_0^{2i-1}\text{Cov}(Y_{i+1}, Y_n)}{\sqrt{1 - \rho_0^{2i}}}$$

$$= \frac{\rho_0^{2n-2i} - \rho_0^{2i-1} \rho_0^{2n-2i}}{\sqrt{1 - \rho_0^{2i}}}$$

$$= 0.$$

For any $1 \leq i < j < n$, there is

$$\text{Cov}(Z_i, Z_j) = \frac{\text{Cov}(Y_i, Y_j) - \rho_0^{2i-1}\text{Cov}(Y_{i+1}, Y_j) - \rho_0^{2j-1}\text{Cov}(Y_i, Y_{j+1}) + \rho_0^{2i-1+2j-1}\text{Cov}(Y_{i+1}, Y_{j+1})}{(1 - \rho_0^{2i})(1 - \rho_0^{2j})}$$

$$= \frac{\rho_0^{2j-1-2i-1} - \rho_0^{2i-1-2i} - \rho_0^{2j-1-2i} + \rho_0^{2i-1+2j-1} \rho_0^{2i-2j}}{(1 - \rho_0^{2i})(1 - \rho_0^{2j})}$$

$$= 0.$$

Those are enough to conclude that the covariance matrix of $z$ is the identity matrix. ◊

**Lemma 2.** Let

$$S_n = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} Z_i Z_j,$$

where $a_{i,j}$ only depends on $(i, j)$ with $a_{1,1} > 0$ and $Z_1, \cdots, Z_n$ are iid $N(0, 1)$. If there exists a positive $C$ such that

$$|a_{i,j}| \leq C/(i + j)^5,$$  \hspace{1cm} (8)

then for any $M \in \mathbb{R}$ there exists a positive constant $\eta$ such that for any $n$

there is

$$P(S_n \geq M) \geq \eta.$$  \hspace{1cm} (9)
Proof: Let $\phi$ and $\Phi$ be the PDF and CDF of $N(0, 1)$, respectively. There is

$$\lim_{x \to \infty} \frac{2[1 - \Phi(x)]}{\phi(x)} = \lim_{x \to \infty} \frac{2}{x} = 0.$$  

Then, there is a positive constant $a$ such that $P(|Z| \geq x) \leq \phi(x)$ for any $x \geq a$. Without loss of generality, we can choose $a \geq 2$. Let

$$\tilde{S}_n = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|i j.$$  

Then,

$$\tilde{S}_n \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{C_{ij}}{(i+j)^5} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{C}{(i+j)^3},$$

which implies that $\tilde{S}_n$ converges as $n \to \infty$. Let $\tilde{S}_{m,n} = \tilde{S}_n - \tilde{S}_m$ for $m \leq n$. Then, we can find an $m_0 \geq a$ such that $\tilde{S}_{m,n} \leq 1$ for any $n > m \geq m_0$. Then,

$$P(S_n \geq M) \geq P(S_m \geq M, |Z_{m_0+1}| \leq m_0 + 1, \ldots, |Z_n| \leq n)$$

$$\geq P(S_{m_0} \geq M + \tilde{S}_{m_0,n}, |Z_{m_0+1}| \leq m_0 + 1, \ldots, |Z_n| \leq n)$$

$$= P(S_{m_0} \geq M + 1) \prod_{i=m_0+1}^{n} P(|Z_i| \leq i)$$

$$\geq P(S_{m_0} \geq M + 1) \prod_{i=m_0+1}^{n} (1 - \phi(i))$$

$$\geq P(S_{m_0} \geq M + 1)[1 - \sum_{i=m_0+1}^{n} \phi(i)]$$

$$\geq P(S_{m_0} \geq M + 1)(1 - b),$$

where $b = \sum_{i=m_0+1}^{\infty} \phi(i)$ satisfying $0 < b < 1$. It is only needed to show that $P(S_{m_0} \geq M + 1) \geq 0$ for any $M \in \mathbb{R}$. Since $m_0$ is fixed and $a_{i,j}$ does not depend on $n$, the distribution of $S_{m_0}$ does not depend on $n$ as $n \to \infty$. Therefore, $P(S_{m_0} \geq M + 1)$ is a constant. Based on the property of the multivariate normal distribution, it is enough to show that $S_{m_0} \geq M + 1$ is not empty for any $M \in \mathbb{R}$. If we choose $Z_2 = Z_3 = \cdots = Z_{m_0} = 0$, then $S_{m_0} = a_{1,1}Z_1^2$. We can find the value of $Z_1$ such that $S_{m_0} \geq M + 1$ since
\( \alpha_{1,1} > 0. \) Therefore, we can choose \( \eta = P(S_m \geq M + 1)(1 - b) \) in (9) to ensure the equation holds. \( \diamond \)

**Theorem 1.** Let \( \rho_0 \in (0, 1) \) be the true parameter of \( \rho \) in Model (5). Then for any \( \rho \in (0, 1) \) and any \( M \in \mathbb{R} \) with \( \rho < \rho_0 \), there exists a positive \( \eta \) such that for any \( n \) there is

\[
P[\ell(\rho) - \ell(\rho_0) \geq M] \geq \eta. \tag{10}
\]

**Proof:** Let \( u_n(\rho) = -(1/2) \sum_{i=1}^{n-1} \log(1 - \rho^{2^i}) \) for \( n > 1 \). Then \( u_n(\rho) \) uniformly converges on \( [a, b] \subseteq (0, 1) \) for any \( 0 < a < b < 1 \). Therefore, \( u_n(\rho) \) is uniformly bounded on \( [a, b] \). Let \( M_1 = \sup_{\rho \in [a, b]} |u_n(\rho)| \). Then, \( M_1 \) is finite. Without loss of generality, we can assume \( \rho, \rho_0 \in [a, b] \). Based on the expression of \( \ell(\rho) \) given by Equation (7), there is

\[
\ell(\rho) - \ell(\rho_0) = u_n(\rho) - [u_n(\rho) - u_n(\rho_0)]
\]

\[
= -\frac{1}{2} \sum_{i=1}^{n-1} [Y_i^2(\frac{\rho_0^{2^i}}{1 - \rho_0^{2^i}} - \frac{\rho^{2^i}}{1 - \rho^{2^i}}) - Y_iY_{i+1}(\frac{2\rho^{2^i-1}}{1 - \rho^{2^i}} - \frac{2\rho_0^{2^i}}{1 - \rho_0^{2^i}})]
\]

\[
+ Y_{i+1}^2(\frac{\rho_0^{2^i}}{1 - \rho_0^{2^i}} - \frac{\rho^{2^i}}{1 - \rho^{2^i}})
\]

\[
= -\frac{1}{2} y^T B y,
\]

where \( y = (Y_1, \cdots, Y_n) \) and the \((i, j)\)-th entry of \( B = B_{\rho, \rho_0} \) is

\[
b_{1,1}(\rho, \rho_0) = \frac{\rho_0^{2^2}}{1 - \rho_0^{2^2}} - \frac{\rho^{2^2}}{1 - \rho^{2^2}},
\]

\[
b_{i,i}(\rho, \rho_0) = (\frac{\rho_0^{2^i}}{1 - \rho_0^{2^i}} - \frac{\rho^{2^i}}{1 - \rho^{2^i}}) - (\frac{\rho_0^{2^{i-1}}}{1 - \rho_0^{2^{i-1}}} - \frac{\rho^{2^{i-1}}}{1 - \rho^{2^{i-1}}}), 1 < i < n,
\]

\[
b_{n,n}(\rho, \rho_0) = \frac{\rho_0^{2^n}}{1 - \rho_0^{2^n}} - \frac{\rho^{2^n}}{1 - \rho^{2^n}},
\]

\[
b_{i,i+1}(\rho, \rho_0) = \frac{\rho_0^{2^{i-1}}}{1 - \rho_0^{2^{i-1}}} - \frac{\rho^{2^{i-1}}}{1 - \rho^{2^{i-1}}}, 1 \leq i < n,
\]

with \( b_{i,i}(\rho, \rho_0) = b_{i,j}(\rho, \rho_0) \) and \( b_{i,j}(\rho, \rho_0) = 0 \) if \( |i - j| > 1 \). Let \( Z_i = (Y_i - \rho_0^{2^{i-1}}Y_{i+1})/\sqrt{1 - \rho_0^{2^i}} \) for \( 1 \leq i < n \) and \( Z_n = Y_n \). Then, \( Z_1, \cdots, Z_n \) are...
iid $N(0,1)$ according to Lemma 1 and

$$Y_n = Z_n$$

$$Y_i = \sqrt{1 - \rho_0^2} Z_i + \rho_0^{n-1-2^{i-1}} \left( \prod_{k=1}^{n-i} \sqrt{1 - \rho_0^{2^k-1}} \right) Z_n$$

$$+ \sum_{l=i+1}^{n-1} \rho_0^{2^l-2^{l-1}} \left( \prod_{k=1}^{i-l} \sqrt{1 - \rho_0^{2^k-1}} \right) Z_l, 1 \leq i < n.$$ Based on $z = (Z_1, \cdots, Z_n)$, we can write

$$\ell(\rho) - \ell(\rho_0) - [u_n(\rho) - u_n(\rho_0)] = -\frac{1}{2} z^T A z,$$

where

$$z^T A z = \sum_{i=1}^n b_{i,i}(\rho, \rho_0) Y_i^2 + \sum_{i=1}^{n-1} 2b_{i,i+1}(\rho, \rho_0) Y_i Y_{i+1}.$$ Let $a_{i,j}(\rho, \rho_0)$ be the $(i, j)$th entry of $A$ for $i, j = 1, \cdots, n$. Then, we can compute $a_{i,j}(\rho, \rho_0)$. The results are

$$a_{i,i}(\rho, \rho_0)$$

$$= b_{i,i}(\rho, \rho_0) (1 - \rho_0^{2^i}) + \sum_{l=1}^{i-1} b_{l,i}(\rho, \rho_0) \rho_0^{2^l-2^{l-1}} \left( \prod_{k=1}^{i-l} \sqrt{1 - \rho_0^{2^k-1}} \right) \left( \prod_{k=1}^{i-l} \sqrt{1 - \rho_0^{2^k+k}} \right)$$

$$+ 2 \sum_{l=1}^{i-2} b_{l,i+1}(\rho, \rho_0) \rho_0^{2^l-2^{l-1}} \left( \prod_{k=1}^{i-l} \sqrt{1 - \rho_0^{2^k-1}} \right) \left( \prod_{k=1}^{i-l} \sqrt{1 - \rho_0^{2^k+k}} \right)$$

$$+ 2b_{i-1,i}(\rho, \rho_0) \rho_0^{2^{i-2}} \sqrt{(1 - \rho_0^{2^i})(1 - \rho_0^{2^{i-1}})}, 1 \leq i < n,$$

$$a_{n,n}(\rho, \rho_0)$$

$$= b_{n,n}(\rho, \rho_0) + \sum_{l=1}^{n-1} b_{l,n}(\rho, \rho_0) \rho_0^{2^n-2^l} \left( \prod_{k=1}^{n-l} \sqrt{1 - \rho_0^{2^k-1}} \right) \left( \prod_{k=1}^{n-l} \sqrt{1 - \rho_0^{2^k+k}} \right)$$

$$+ 2 \sum_{l=1}^{n-2} b_{l,n+1}(\rho, \rho_0) \rho_0^{2^n-2^{l-1}} \left( \prod_{k=1}^{n-l} \sqrt{1 - \rho_0^{2^k-1}} \right) \left( \prod_{k=1}^{n-l} \sqrt{1 - \rho_0^{2^k+k}} \right)$$

$$+ 2b_{n-1,n}(\rho, \rho_0) \sqrt{1 - \rho_0^{2^{n-1}}},$$
\(a_{i,j}(\rho, \rho_0)\)

\[= b_{i,i}(\rho, \rho_0)\sqrt{1 - \rho_0^{2^i}} \left(\prod_{k=1}^{j-i} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ \sum_{l=1}^{i-1} b_{i,l}(\rho, \rho_0)\rho_0^{2^{i-l} + 2^{j-1} - 2^l} \left(\prod_{k=1}^{j-l} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ \sum_{l=1}^{i-1} b_{i,l+1}(\rho, \rho_0)\rho_0^{2^{i-l} + 2^{j-1} - 2^l} \left(\prod_{k=1}^{j-l-1} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ \sum_{l=1}^{i-1} b_{i,l+1}(\rho, \rho_0)\rho_0^{2^{i-l} + 2^{j-1} - 2^l} \left(\prod_{k=1}^{j-l-1} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ b_{i,i+1}(\rho, \rho_0)\rho_0^{2^{i-1} - 2^i} \left(\prod_{k=1}^{j-i} \sqrt{1 - \rho_0^{2^k}}\right), 1 \leq i < j < n,\]

and

\(a_{i,n}(\rho, \rho_0)\)

\[= b_{i,i}(\rho, \rho_0)\rho_0^{n-1 - 2^i} \left(\prod_{k=1}^{n-i} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ \sum_{l=1}^{i-1} b_{i,l}(\rho, \rho_0)\rho_0^{n-1 + 2^{i-1} - 2^l} \left(\prod_{k=1}^{n-l} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ \sum_{l=1}^{i-1} b_{i,l+1}(\rho, \rho_0)\rho_0^{n-1 + 2^{i-1} - 2^l} \left(\prod_{k=1}^{n-l-1} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ \sum_{l=1}^{i-1} b_{i,l+1}(\rho, \rho_0)\rho_0^{n-1 + 2^{i-1} - 2^l} \left(\prod_{k=1}^{n-l-1} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ I_{i\neq n-1} b_{i,i+1}(\rho, \rho_0)\rho_0^{n-1 - 2^i} \left(\prod_{k=1}^{n-i-1} \sqrt{1 - \rho_0^{2^k}}\right)\]

\[+ I_{i=n-1} b_{n-1,n}(\rho, \rho_0)\sqrt{1 - \rho_0^{2^{n-1}}}, 1 \leq i < n.\]

Note that \(b_{i,j}(\rho, \rho_0)\) does not depend on \(n\). Observing the terms above, we find that: (a) \(a_{i,j}(\rho, \rho_0)\) does not depend on \(n\) if \(i, j \neq n\); (b) \(b_{i,j}(\rho, \rho_0)\) goes to zero exponentially fast as \(i\) and \(j\) increase if \(0 \leq \rho, \rho_0 < 1\); and (c)
$$a_{1,1}(\rho, \rho_0) = \rho_0^2 - \rho^2(1 - \rho_0^2)/(1 - \rho^2)$$ is positive if $\rho < \rho_0$ and is negative if $\rho > \rho_0$. Thus, $a_{i,j}(\rho, \rho_0)$ satisfies Conditions (8). This is enough to draw Equation (10) according to Lemma 2.

**Corollary 1.** For any $\rho, \rho_0 \in (0, 1)$ with $\rho < \rho_0$, $\ell(\rho) - \ell(\rho_0)$ goes to a non-degenerate random variable in distribution, which is larger than any positive number with a positive probability.

*Proof:* Note that $\ell(\rho) - \ell(\rho_0)$ is a Cauchy sequence in probability. We can find a random variable $X$ such that $\ell(\rho) - \ell(\rho_0) \overset{D}{\to} X$ as $n \to \infty$. According to Theorem 1, for any $M > 0$ there is an $\eta > 0$ such that $P(X > M) \geq \eta$, which is the conclusion of the Corollary.

**Theorem 2.** If $s_i = 2^{i-1}$ for $i = 1, \cdots, n$ in Model (3), then $\hat{\rho}$ is an inconsistent estimator of $\rho$, which implies that the MLE of $\theta$ is also an inconsistent estimator of $\theta$.

*Proof:* The conclusion can be directly implied by Corollary 1.

3. Discussion

We have constructed a special example in which the MLE of covariance parameters is inconsistent under the framework of increasing domain asymptotics. The example may be classified as an infinitely sparse case about the pattern of irregularly spaced locations in spatial statistics. Comparing to the case considered under the framework of the infill asymptotics which assumes the spaced locations are infinitely dense, the example of the infinitely sparse case provides another way to understand the impact of location patterns on asymptotic properties of estimators of covariance parameters in a spatial statistics. Therefore, it is also necessary to consider this case in the understanding of the asymptotic behavior of estimators of covariance parameters.

**Reference**


