Chapter 2: Probability

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Chapter Overview

This chapter focuses on "Probability"

- Random Experiments, Sample Space and Events.
  - Random Experiment (R.E.)
  - Sample Space
  - Events
  - Examples
- Introduction of probability
  - Axiom
  - Properties
  - Examples
- Counting Techniques
  - Combinations
  - Permutations
- Conditional Probability
  - Definition
  - Related rules and theorem
- Independence

Random Experiment, Sample Space & Events

- **Random Experiment (R.E.)**
  A R.E. is any action or process whose outcome is subject to uncertainty. i.e.,many outcomes, each has a certain chance to happen.
- **Sample Space of a R.E.**
  Denoted by $\mathcal{S}$, is the set of all possible outcomes of the R.E.
- **Event of a R.E.**
  Denote by capital letters (event A, B, C etc), an event is any collection of outcomes contained in the sample space $\mathcal{S}$ of the R.E..
  - An event is said to be **simple** if it contains of exactly one outcome.
  - An event is said to be **compound** if it contains more than one outcome.

R.E. Example 2.1.1

Example 2.1.1 – This example is not in your textbook.

R.E.: Toss a die and look at the outcome of the die.
Sample Space: $\{1, 2, 3, 4, 5, 6\}$
Events:
$A =$out come is 2 $=\{2\}$ simple event
$B =$outcome is greater than 3 $= \{4, 5, 6\}$ compound event

R.E. Example 2.1.2

Example 2.1.2 – This example is not in your textbook.

R.E.: Toss a fair coin until a head appears.
Sample Space: $\{H, TH, TTH, TTTH, TTTTH, ...\}$
Events:
$A =$ exactly 2 tosses $= \{TH\}$ simple event
$B = <4$ tosses $= \{H, TH, TTH\}$ compound event
$C = \geq 2$ and ≤ 5 tosses $= \{TH, TTH, TTTH, TTTTH\}$ compound event
$D = >3$ tosses $= \{TTTH, TTTTH, TTTTTH, ...\}$ compound event

Using Set Theory Results

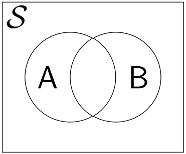
An even is a set of outcomes, we may use some set theory results to form new events.

- **Union** $A \cup B$:
  is the event containing all outcomes in A or B.
- **Intersect** $A \cap B$**:**
  is the event containing all outcomes in both A and B.
- **Compliment of A** A^c**:**
  A^c is the event containing all outcomes in the sample space $\mathcal{S}$ that are not contained in A. We have $A \cup A^c = \mathcal{S}$.
- **Mutually Exclusive (disjoint):**
  A and B are said to be mutually exclusive if they contain no common outcome. i.e. $A \cap B = \emptyset$. Here $\emptyset$ denotes an empty set.

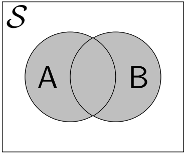
Venn Diagram

Venn diagram is a visual representation of events and their operations.

- Venn diagram of two events A and B

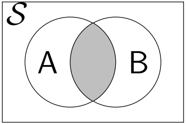


- Venn diagram of $A \cup B$ (shaded region)

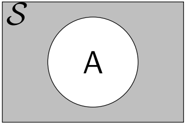


Venn Diagram Continued

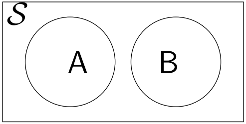
- Venn diagram of $A \cap B$ (shaded region)



- Venn diagram of A^c (shaded region)



- Venn diagram of mutually exclusive A and B



Set Operations Example 2.1.2

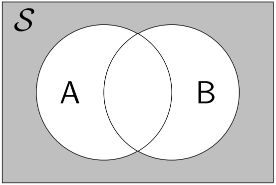
Example 2.1.2

$A =$ exactly 2 tosses $= \{TH\}$
$B = <4$ tosses $= \{H, TH, TTH\}$
$C = \geq 2$ and ≤ 5 tosses $= \{TH, TTH, TTTH, TTTTH\}$
$D = >3$ tosses $= \{TTTH, TTTTH, TTTTTH, ...\}$

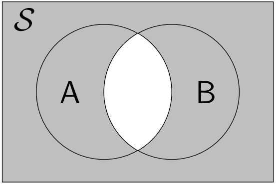
$A \cup B \cup C = \{H, TH, TTH, TTTH, TTTTH\}$
$B \cap C = \{TH, TTH\}$
$C \cap D = \{TTTH, TTTTH\}$
$B^c = \{TTTH, TTTTH, ...\} = D$

DeMorgan's Law

$$(A \cup B)^c = A^c \cap B^c$$



$$(A \cap B)^c = A^c \cup B^c$$



Probability

Probability is a concept that can be used to model randomness of events. For a given event A, denote the probability of A by $P(A)$.

Axiom of probability:

1. For any given event A, $P(A) \geq 0$.
2. $P(\mathcal{S}) = 1$.
3. If $A_1, A_2, A_3, \ldots$ is a countably infinite collection of mutually exclusive events, then:

$$P(A_1 \cup A_2 \cup A_3 \ldots.) = \sum_{i=1}^{\infty} P(A_i)$$

This indicates that, if we have a finite collection of mutually exclusive events $A_1, A_2, \ldots., A_k$, then:

$$P(A_1 \cup A_2 \cup A_3 \ldots A_k) = \sum_{i=1}^{k} P(A_i)$$

Properties of Probability

Proposition

1. *For any event* A, $P(A) = 1 - P(A^c)$.
2. *Probability of the empty set:* $P(\emptyset) = 0$.
3. *For mutually exclusive events,* $P(A \cap B) = P(\emptyset) = 0$.
4. *For any two events* A *and* B,
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Probability Example 2.1.3

Example 2.1.3 – This example is not in your textbook.

A card is draw from a deck of 52 playing cards. Let event A be: the card is ace. Let event B be: the card is diamond. Now what is the probability that the card is ace or diamond?

We are actually looking for
$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A) = \frac{4}{52}$, $P(B) = \frac{13}{52}$, $P(A \cap B) = P(\text{diamond ace}) = \frac{1}{52}$
$P(A \cup B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$

Counting Techniques

- Product Rule
  - Product rule for ordered pairs
  - Product rule for k-tuples
- Permutations
  - Example
  - Permutation formula
- Combinations
  - Example
  - Combination formula
- Use of counting techniques
  - Examples

Equally Likely Outcomes

- When outcomes of a R.E. are equally likely,probability of an event A could be calculated as:

$$P(A) = \frac{N(A)}{N}$$

  Here $N(A)$ is the number of outcomes contained in event A, and N is the number of outcomes in the sample space.

- Example: draw a card from a deck of 52 cards. Let event A be the card is ace.

$$N(A) = 4, N = 52, P(A) = \frac{4}{52}.$$

- Thus, calculation of probability of some events becomes:
  - Construct a sample space with equally likely outcomes.
  - Count $N(A)$.
  - $P(A) = \frac{N(A)}{N}$

Product Rule for Ordered Pairs

- An ordered pair refers to a pair of objects O_1, O_2 and (O_2, O_1) is considered different than (O_1, O_2). i.e., order matters.

Proposition

If the first element or object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is $n_1 n_2$.

Example 2.2.1 (not in textbook) Two bags, bag #1 contains two balls marked 1 and 2 respectively; bag #2 contains three balls marked 1, 2 and 3 respectively. First pick one ball from bag #1, and then pick one ball from bag #2, how many different possibilities will we get?

$(1,1),(1,2),(1,3),(2,1),(2,2),(2,3).$
bag #1 can be selected in 2 ways, $n_1 = 2$, bag #2 can be selected in 3 ways, $n_2 = 3$, so the number of pairs is $n_1 n_2 = 2 \cdot 3 = 6$.

Product Rule for k-Tuples

- A k-tuple is an ordered collection of k objects.

Proposition

A ordered collections of k elements (k-tuple), n_1 choices for the first element; n_2 choices for the second element,...; n_k choices of the kth element. Then there are $n_1 n_2 \ldots n_k$ possible k-tuples.

Example 2.2.2 (not in textbook) Three bags, bag #1 contains two balls marked 1 and 2 respectively; bag #2 contains three balls marked 1, 2 and 3 respectively; bag #3 contains 3 balls marked a, b, c, respectively. First pick one ball from bag #1, and then pick one ball from bag #2, then pick one from bag #3. How many different pairs will we get?

(1,1,a),(1,1,b), (1,1,c), (1,2,a), (1,2,b), (1,2,c), (1,3,a), (1,3,b),(1,3,c), (2,1,a),(2,1,b), (2,1,c), (2,2,a), (2,2,b), (2,2,c), (2,3,a), (2,3,b),(2,3,c) We are looking at a 3-tuple. Bag #1 can be selected in 2 ways, $n_1 = 2$, bag #2 can be selected in 3 ways, $n_2 = 3$, bag #3 is selected in 3 ways, $n_3 = 3$, so the number of 3-tuples is $n_1 n_2 n_3 = 2 \cdot 3 \cdot 3 = 18$.

Permutations

Definition

Any ordered sequence of k objects taken from a set of n distinct objects is called a **permutation** of size k. The number of permutations of size k constructed from n objects is denoted by $P_{k,n}$.

Example 2.2.3(not in textbook) One suit of 13 cards, all spades. Pick 4 out of these 13 card, how many different permutations? How about picking 5, and 13?

4 cards: 1st card 13 choices, 2nd card 12 choices, 3rd card 11 choices, 4th card, 10 choices. So $13 \cdot 12 \cdot 11 \cdot 10$
5 cards: $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$
13 cards: $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 ... 2 \cdot 1 = 13!$.

Permutation Formula

Formula for $P_{k,n}$

$$P_{k,n} = n(n-1)(n-2)....(n-k+1)$$

So,

$$P_{k,n} = \frac{n(n-1)...(n-k+1)(n-k)(n-k-1)....2 \cdot 1}{(n-k)(n-k-1)...2 \cdot 1}$$

i.e.,

$$P_{k,n} = \frac{n!}{(n-k)!}$$

Combinations (Unordered Sequence)

Definition

Any unordered subset of size k from n distinct objects is called a **combination**. The number of combinations of size k from n objects is denoted by $C_{k,n}$ or $\binom{n}{k}$. We will use $\binom{n}{k}$.

Example 2.2.4(not in textbook) 5 balls marked 1,2,3,4,5, respectively. Take out 3 balls, how many different combinations?

If we were to calculate the number of permutations, we have $P_{3,5} = 5 \cdot 4 \cdot 3$. But notice now order no longer matters, so any permutation of the same collection of balls should be counted as the same combination. There are 3! permutations of 3 balls, so, the number of combinations $\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!}$=10

Combination Formula

Formula for $\binom{n}{k}$

$$\binom{n}{k} = \frac{P_{k,n}}{k!}$$

So,

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!}$$

same as,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Use Counting Techniques

The general steps include:

- Construct the sample space where each outcome are equally likely to appear.
- Find out the event of interest.
- Count the outcomes in the event
- Find probability by $\frac{N(event)}{N}$

Examples

Let's look at how to use these counting techniques to calculate probability (not in textbook)

- **Example 2.2.5** Toss 2 dice, what is the probability that the sum of the outcomes of the two tosses is 5?
- **Example 2.2.6** Draw two cards from a suit of 13 cards (say, heart), what is the probability that the sum of two cards is even? (A=1, J=11, Q=12, K=13).
- **Example 2.2.7** Draw two cards from a deck of 52 cards, what is the probability that the two cards are both 3?
- **Example 2.2.8** Draw two cards from a deck of 52 cards, what is the probability that the two cards are of the same number?
- **Example 2.2.9** Draw two cards from a deck of 52 cards, what is the probability that the two cards are not of the same number or they are both not hearts?

Conditional Probability

Example 2.3.1

Draw a card from a deck of 52 cards, what is the probability of getting a heart? Suppose now we already know that the card not club, what is the probability of getting a heart?

When we don't know anything, the probability is $\frac{13}{52}$; given the information that the card is not club, the probability becomes $\frac{1}{3}$.

Conditional probability is used to model the probability of events when certain information is provided.

Definition

Two events A and B with $P(B) > 0$, the **conditional probability of** A **given** B **has occurred** is denoted as $P(A|B)$,and is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Rearranging the terms, we get the **product rule for** $P(A \cap B)$:

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Conditional Probability Examples

Example 2.3.1 (not in textbook) Let event $A =$ getting a heart, event $B =$ getting a 4. What is the conditional probability of A given B?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/52}{4/52} = \frac{1}{4}$$

Example 2.3.2 (not in textbook) Deal 2 cards from a deck of 52 cards (well shuffled, without replacement). Find the probability that they are both hearts. How about if we deal 3, and there are 2 hearts and a club?

$P(\text{1st H}) = \frac{13}{52}$, $P(\text{2nd H}|\text{1st H}) = \frac{12}{51}$, so
$P(\text{1st H and 2nd H}) = \frac{13}{52} \cdot \frac{12}{51}$ Deal 3, we have
$P(\text{2H \& 1C}) = \binom{3}{1} \frac{13}{52} \frac{12}{51} \frac{13}{50}$

Bayes' Theorem

Theorem (Law of total probability)

Let $A_1, ...A_k$ be mutually exclusive and $A_1 \cup A_2.... \cup A_k = \mathcal{S}$. Then for any other event B,

$$P(B) = P(B|A_1)P(A_1) + ... + P(B|A_k)P(A_k) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

Theorem (Bayes' theorem)

Let $A_1, A_2, ..., A_k$ be a collection of mutually exclusive events with $P(A_i) > 0$ for $i = 1, ..., k$, and $A_1 \cup A_2.... \cup A_k = \mathcal{S}$. Then for any other event B with $P(B) > 0$,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}$$

Bayes' Theorem Examples

- **Example 2.3.3**(not in textbook) Test for diabetes, suppose we know that 5% of the total population has diabetes. If a person is sick, diabetes test successfully test it with 95% correct rate, and fails 5% of the time. If a person is not sick, 1% of the time the test will misjudge. What is the probability that when the test shows a person is sick, the person is really sick?
- **Example 2.3.4**(not in textbook) Flip a coin. If head, draw a ball from a bag with 2 white and 2 black balls; if tail, draw a ball from a bag with 1 white and 2 black balls. Now if the ball is white, what is the probability that the coin was a head?

Independence of Events

Definition (Independence between two events A and B)

A, B are independent if $P(A|B) = P(A), P(B) \neq 0$.

i.e., knowing B happening does not affect the possibility of A.

Definition (Alternative definition)

A, B are independent if $P(A \cap B) = P(A)P(B)$.

Easy to show that these two are equivalent using $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

Definition (Mutually independence)

Events $A_1, A_2, ..., A_n$ are said to be mutually independent if for every $k, (k = 2, 3, ..., n)$ and every subset of indices $i_1, i_2, ..., i_k$,

$$P(A_{i_1} \cap A_{i_2} ... \cap A_{i_k}) = P(A_{i_1})P(A_{i_2})...P(A_{i_k})$$

i.e., any subset of events are mutually independent.

Independence and Mutually Exclusiveness

- Independence $\neq$ mutually exclusiveness
- Independence indicates the events knowing one does not give more information about the other (in terms of probability)
- Mutually exclusiveness indicates no common outcome(s).

Example 2.3.5 Deal a card, event $A =$ card is heart, event $B =$ card is not heart. A, B independent? Mutually exclusive?

Independence Examples

- **Example 2.3.6** Deal two cards, without replacement, event A = 1st card is heart, B = 2nd is heart. Is A, B independent? event C = 2nd is club, is A, C independent?
- **Example 2.3.7** All components in the network functions independently. $P(A) = 0.90$, $P(B) = 0.92$, $P(C) = 0.80$, $P(D) = 0.95$, what is the probability that the network will fail?