1. (15 points) A life of a lightbulb (in hours) is exponentially distributed. A pack of 4 lightbulbs can come from either manufacturer A or B. Bulbs from A have an average life expectancy of 2000 hours, lights from B 3000 hours, respectively. From the pack of 4, two lightbulbs burned out in less than 2500 hours, and two lasted longer than 2500 hours. What is the probability that the pack was manufactured by A?

Solution: Let \( E \) denote the event that two lightbulbs out of the pack lasted less than 2500 hours, and two lasted longer than 2500 hours. Using Bayes’ Rule,

\[
P(A|E) = \frac{P(A)P(E|A)}{P(E)} = \frac{P(A)P(E|A)}{P(A)P(E|A) + P(B)P(E|B)}
\]

Since we are not given any additional information about manufacturers A and B, we assume that the probabilities that the pack comes from A or from B are the same, \( \frac{1}{2} \). Given a manufacturer, the events of different lightbulbs lasting less or longer than 2500 hours are conditionally independent, so the variable indicating the number of lightbulbs that burn out before 2500 hours (given a manufacturer) is binomially distributed. What is left is to find the probability of “success”, burning out before 2500 hours, for each manufacturer.

For exponentially distributed random variables, the expected value is equal to the inverse of the exponential parameter \( \lambda \), so \( \lambda_A = \frac{1}{2000} \) and \( \lambda_B = \frac{1}{3000} \). Now let \( X \) be a random variable denoting the life in hours of a lightbulb.

\[
p_A = P(X < 2500|A) = 1 - e^{-\frac{2500}{2000}} = 1 - e^{-1.25} = 0.7135;
\]

\[
p_B = P(X < 2500|B) = 1 - e^{-\frac{2500}{3000}} = 1 - e^{-\frac{5}{6}} = 0.5654;
\]

\[
P(A|E) = \frac{0.5 \times \binom{4}{2}0.7135^2 \times 0.2865^2}{0.5 \times \binom{4}{2}0.7135^2 \times 0.2865^2 + 0.5 \times \binom{4}{2}0.5654^2 \times 0.4346^2} = 0.4090.
\]

2. (10 points) A probability density function for a random variable \( X \) is given by

\[
f(x) = \begin{cases} a + bx^3 & : x \in [0, 1], \\ 0 & : \text{otherwise}. \end{cases}
\]

Given that \( E[X] = 0.5 \), find \( a \) and \( b \).

Solution: First, the probability density function has to integrate to 1:

\[
1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{1} (a + bx^3) \, dx = \left[ ax + \frac{b}{4}x^4 \right]_{0}^{1} = a + \frac{b}{4} \implies a = 1 - \frac{b}{4}.
\]
Second, the expected value is provided:

\[
0.5 = E[X] = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{1} (ax + bx^4) \, dx = \left( \frac{a}{2} x^2 + \frac{b}{5} x^5 \right) \bigg|_{0}^{1} = \frac{a}{2} + \frac{b}{5} = \frac{1}{2} \left( 1 - \frac{b}{4} \right) + \frac{b}{5};
\]

\[
= \frac{1}{2} - \frac{b}{8} + \frac{b}{5} = \frac{1}{2} + \frac{3b}{40} \Rightarrow b = 0 \Rightarrow a = 1 - \frac{b}{4} = 1.
\]

3. (10 points) An urn contains 40 white balls and 60 black balls. A ball is drawn with replacement 100 times. What is the probability that a white ball appeared at least 45 times? (Use continuity correction)

**Solution:** Let \( X \) denote the number of white balls from the 100 draws. \( X \) is a binomial random variable with \( N = 100 \) and \( p = 0.4 \).

\[
P(X \geq 45) = P(44.5 \leq X \leq 100.5) = P \left( \frac{44.5 - 100 \times 0.4}{\sqrt{100 \times 0.4 \times 0.6}} \leq X \leq \frac{100.5 - 100 \times 0.4}{\sqrt{100 \times 0.4 \times 0.6}} \right)
\]

\[
\approx \Phi(12.3495) - \Phi(0.9186) = 0.1792.
\]

4. (15 points) If \( X \) is a normal variable with mean 2 and variance 9, find the probability density function for \( Y = -X^3 \).

**Solution:** \( Y = g(X) = -X^3 \). Thus \( X = g^{-1}(Y) = -\sqrt[3]{Y} \). The probability density function for \( X \) is

\[
f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-2}{3})^2}.
\]

Then

\[
f_Y(y) = f_X(g^{-1}(Y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{2-y}{3})^2} \times \frac{1}{3} |y|^{-\frac{2}{3}}.
\]

The domain for \( f_Y \) is the range of the transformation function \( g \). In this case, the range is \( \mathbb{R} \), so \( y \in \mathbb{R} \).