STAT/MATH 416 Fall 09
Practice Final Exam
December 6, 2009

You are not allowed to use books or notes. Non-programmable calculators approved for 1/P exam are permitted. Please read the directions carefully. The exam is graded out of 350 points with a possibility of additional 100 points for solving extra credit problems. No partial credit will be given for extra credit problems. You have 120 minutes to complete the exam. Standard normal table is provided on the last page. **Please show all your work.**

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1. (25 points) Suppose $A$, $B$, and $C$ are events such that $P(A \cup B) = 0.55$, $P(A \cup C) = 0.6$, $P(A \cup (B \cap C)) = 0.4$, and $P(A) = 0.2$. What is $P(A \cup B \cup C)$?

2. (20 points) Two urns contain identical red and blue balls. The first urn has 5 of each while the second contains 15 red balls and an unknown number of blue balls. The first urn is chosen with probability 0.4, the second with probability 0.6, and one ball is drawn from the chosen urn. If the probability of drawing a blue ball is 0.35, how many blue balls are in the second urn?
3. (a) (25 points) A coin is flipped repeatedly until the first heads appears. What is the probability of getting heads on any given flip if the probability that the coin is going to be flipped more than twice is 0.64?

(b) (15 points) The same coin is flipped repeatedly until heads appears for the 10th time. Find the probability that this would happen on the 45th flip.
4. (25 points) A fair coin is flipped 7200 times, and a fair die is rolled 1000 times. What is the probability that the number of heads is larger than the sum of all die roll values?

5. (20 points) The number of tornadoes hitting the county in a year is Poisson distributed. In a given year, the county is twice as likely to get hit by exactly 2 tornadoes than by exactly 4. What is the probability that more than 1 tornado would form over the county?
6. A lifetime of an oil filter is exponentially distributed with the expected value of 150 days.

(a) (20 points) If an oil filter did not need to be replaced for 120 days, what is the probability that it will need to be replaced by 180 days?

(b) (25 points) A lifetime of an air filter is exponentially distributed with the expected value of 180 days. A new oil filter and an air filter have just been put in. If they wear out independently of one another, what is the probability that an air filter will need to be replaced more than 10 days after an oil filter?
7. (20 points) The length of a person’s step is uniformly distributed between 50 and 100 cm. What is the probability that the person would walk less than 740m in 1000 steps? (Assuming that the lengths of the steps are independent of one another.)

8. 10 identical dice lay on the table. 6 of them are fair, and the remaining 4 are loaded with the face 1 coming up half the time, and the remaining faces coming up with the same frequency. One die is chosen at random. It has been rolled ten times, and the face 6 came up twice.

   (a) (25 points) Find the probability that a fair die has been chosen.
(b) (15 points, extra credit) Find the probability that the next roll of the same die will show 1.

(c) (15 points, extra credit) Find the expected sum of face values for the next 100 rolls of the same die.
9.

\[ f_{XY}(x, y) = \begin{cases} 
  cxy & x \geq 0, \ y \geq 0, \ x + y \leq 1, \\
  0 & \text{otherwise}.
\end{cases} \]

(a) (15 points) Find \( c \).

(b) (15 points) Find \( f_X(x) \) and \( f_Y(y) \)

(c) (15 points) Find \( E[X], E[Y], \text{Var}(X), \) and \( \text{Var}(Y) \).
\[ f_{XY}(x, y) = \begin{cases} 
  cxy & x \geq 0, \ y \geq 0, \ x + y \leq 1, \\
  0 & \text{otherwise}. 
\end{cases} \]

(d) (20 points) Find \( \text{Cov}(X, Y) \) and \( \rho(X, Y) \).

(e) (10 points) Are \( X \) and \( Y \) independent? Provide a proof for your answer.
10. Batteries produced by a manufacturer A last on average 200 hours while batteries made by B last on average 180 hours. A flashlight has had a new battery put in, with probability 0.6 that it is made by A, and probability 0.4 that it is made by B. Assume that the battery life for each manufacturer is exponentially distributed.

(a) (20 points) What is the expected life of the battery in the flashlight (in hours)?

(b) (15 points, extra credit) What is the standard deviation for the life of the battery in the flashlight (in hours)?
11. Let $X$ be a uniform random variable on $(0, 3)$, and let $Y$ be an exponential random
variable with mean 2. Assume that $X$ and $Y$ are independent.

(a) (15 points) Find the variance of $X + Y$.

(b) (20 points) Let $U = XY$, and let $V = \frac{Y}{X}$. What is the joint probability density
function for $U$ and $V$?
Let $X$ be a uniform random variable on $(0, 3)$, and let $Y$ be an exponential random variable with mean 2. Assume that $X$ and $Y$ are independent.

(c) (20 points, extra credit) Find the probability density function for $X + Y$.

(d) (15 points, extra credit) Find the moment generating function for $X + Y$. 
12. (20 points, extra credit) 50 distinct numbers are chosen at random from \{1, \ldots, 100\}. What is the expected number of pairs of consecutive numbers among these 50 numbers? (For example, \{1, 2, 3\} contains two pairs of consecutive numbers, \{1, 2\}, and \{2, 3\}).