Name	SOLUTION
Student ID #	
Instructor:	Sergey Kirshner

STAT/MATH 416 Fall 09

Practice Midterm #1
September 28, 2009

You are not allowed to use books or notes. Non-programmable calculators approved for 1/P exam are permitted. Please read the directions carefully. The exam is graded out of 175 points with a possibility of additional 20 points for solving an extra credit problem. You have 50 minutes to complete the exam. **Please show all your work.**

Problem #	Number of points
1	/20
2	/20
3	/20
4	/20
5	/30
6	/35
7	/30
8 (extra)	/20
Total	/175

1. (20 points) A fair coin is flipped 10 times. What is the probability that more than two tosses are heads?

   Solution: Let N be the number of heads after 10 tosses.

$$\begin{aligned} P(N>2) &= 1-P(N\leq 2)=1-P(N=0)-P(N=1)-P(N=2) \\ &= 1-\binom{10}{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{10}-\binom{10}{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{9}-\binom{10}{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{8} \\ &= \frac{1024-1-10-45}{1024}=\mathbf{0.9453125}. \end{aligned}$$

2. (20 points) A student is taking midterms in Linear Algebra, Probability, and Analysis. The probabilities of getting an A in Linear Algebra, Probability, and Analysis are 0.3, 0.5, and 0.25, respectively. Given that grades on the midterms in these courses are mutually independent, the probability that a student will get at least one A.

   Solution: Let LA denote the event of getting an A on the Linear Algebra test, P on the Probability test, and A on the Analysis test, respectively. Using DeMorgan's Law and taking independence into account,

$$\begin{aligned} P(LA\cup P\cup A) &= 1-P\left((LA\cup P\cup A)^{c}\right)=1-P\left(LA^{c}\cap P^{c}\cap A^{c}\right) \\ &= 1-P\left(LA^{c}\right)P\left(P^{c}\right)P\left(A^{c}\right) \\ &= 1-(1-P(LA))(1-P(P))(1-P(A)) \\ &= 1-(1-0.3)(1-0.5)(1-0.25)=\mathbf{0.7375}. \end{aligned}$$

3. (20 points) 60% of students are well prepared for the exam, 40% are poorly prepared. A well-prepared student has 0.6 probability of receiving an A on the exam, 0.3 of receiving a B, and 0.1 of receiving a grade less than B. A poorly prepared student has 0.2 probability of receiving an A, 0.2 probability of receiving a B, 0.6 probability of receiving a grade below B. Find the probability that a student was well prepared given that he/she received a B on the exam.

Solution: Let W denote the event that the student is well-prepared, and P that he/she is poorly prepared. Let A denote the event that a student receives and A, B a B, and F for less than a B, respectively. Then using Bayes' Rule

$$\begin{aligned} P(W|B) &= \frac{P(W)\,P(B|W)}{P(B)} = \frac{P(W)\,P(B|W)}{P(W)\,P(B|W) + P(P)\,P(B|P)} \\ &= \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.4 \times 0.2} = \frac{0.18}{0.18 + 0.08} \approx \mathbf{0.6923}. \end{aligned}$$

4. (20 points) In a particular zoo, 20 animals have fur, 25 are carnivorous, and 8 are furry carnivores with natural habitat in Africa. If 35 of zoo's occupants are either carnivorous or furry, how many of its furry carnivores are not naturally found in Africa?

Solution: Let F denote the event that a zoo animal is furry, C carnivorous, and A with habitat in Africa, respectively.

$$\begin{aligned} |F \cap C \cap A^c| &= |F \cap C| - |F \cap C \cap A| \\ &= |F| + |C| - |F \cup C| - |F \cap C \cap A| = 20 + 25 - 35 - 8 = \mathbf{2}. \end{aligned}$$

5. Die A has one yellow and five orange sides, and die B has one orange and five yellow sides. A fair coin is flipped, and die A is selected if it lands heads, and die B is selected if it lands tails. Assume that the selected die was rolled once, and it rolled orange side up.

   (a) (15 points) What is the probability that die A is selected?

   Solution: Let A denote the probability that coin A is selected, and let O_1 denote the event that the outcome is orange for first roll, respectively. Then using Bayes' Rule,

$$\begin{aligned} P(A|O_1) &= \frac{P(A)\,P(O_1|A)}{P(O_1)} = \frac{P(A)\,P(O_1|A)}{P(A)\,P(O_1|A) + P(B)\,P(O_1|B)} \\ &= \frac{\frac{1}{2}\times\frac{5}{6}}{\frac{1}{2}\times\frac{5}{6}+\frac{1}{2}\times\frac{1}{6}} = \mathbf{\frac{5}{6}}. \end{aligned}$$

   (b) (15 points) If the same die is rolled again, what is the probability that the orange side will come up again?

   Solution: Let O_2 denote that the die rolled orange on the second roll. Using the definition of conditional probability, total probability rule along with conditional independence,

$$\begin{aligned} P(O_2|O_1) &= \frac{P(O_1 \cap O_2)}{P(O_1)} = \frac{P(O_1 \cap O_2 \cap A) + P(O_1 \cap O_2 \cap B)}{P(O_1 \cap A) + P(O_1 \cap B)} \\ &= \frac{P(A)\,P(O_1|A)\,P(O_2|A) + P(B)\,P(O_1|B)\,P(O_2|B)}{P(A)\,P(O_1|A) + P(B)\,P(O_1|B)} \\ &= \frac{\frac{1}{2}\times\frac{5}{6}\times\frac{5}{6}+\frac{1}{2}\times\frac{1}{6}\times\frac{1}{6}}{\frac{1}{2}\times\frac{5}{6}+\frac{1}{2}\times\frac{1}{6}} = \mathbf{\frac{13}{18}}. \end{aligned}$$

6. A random variable X has $\mu = E[X] = 0.75$ and $\sigma = SD(X) = 0.5$.

(a) (5 points) What can be said about $P(0 < X < 1.5)$?

Solution: Applying Chebyshev's inequality

$$\begin{aligned} P(0 < X < 1.5) &= P(|X - 0.75| < 0.75) = P\left(|X - \mu| < \frac{0.75}{0.5}\sigma\right) \geq 1 - \frac{1}{1.5^2} \\ &= \frac{5}{9}. \end{aligned}$$

(b) (15 points) Now, suppose you are told that the functional form for a p.d.f. for X is given as

$$f_X(x) = \begin{cases} \frac{3}{c}\left(-ax^2 - 6x + 34\right) & : \ x \in (0, 2), \\ 0 & : \ \text{otherwise}. \end{cases}$$

Find a and c. (Hint: you don't really need both $E[X]$ and $SD(X)$; only $E[X]$ would suffice.)

Solution: To solve for a and c, we need to solve a system of two equations with two unknowns, one equation for $E[X] = 0.75$, and one for total mass equal to 1.

$$\begin{cases} 1 &= \int_0^2 \frac{3}{c}\left(-ax^2 - 6x + 34\right) dx = \frac{3}{c}\left[-\frac{ax^3}{3} - 3x^2 + 34x\right]\Big|_0^2 = -\frac{8a}{c} + \frac{168}{c}, \\ 0.75 &= \int_0^2 \frac{3}{c}\left(-ax^3 - 6x^2 + 34x\right), dx = \frac{3}{c}\left[-\frac{ax^4}{4} - 2x^3 + 17x^2\right]\Big|_0^2 = -\frac{12a}{c} + \frac{156}{c}. \end{cases}$$

From the first equation, $c = 168 - 8a$. Plugging it for c in the second equation, we obtain $0.75(168 - 8a) = -12a + 156$. Thus $126 - 6a = -12a + 156$, and therefore $6a = 30$, so $a = \mathbf{5}$. Plugging it back, we find $c = 168 - 8 \times 5 = \mathbf{128}$.

(c) (10 points) Using function in (b), find distribution function for X, $F_X(x)$. Leave the expression in terms of fractions, but be sure to specify it for *all* values in $\mathbb{R}$.

Solution: Since the density for $x \leq 0$ and for $x \geq 2$ is 0, $F_X(x) = 0$ for $x \leq 0$, and $F_X(x) = 1$ for $x \geq 2$. For $x \in (0,2)$, $F_X(x) = \int_0^x f_X(u)\,du = \frac{3}{128}\left[-\frac{5}{3}u^3 - 3u^2 + 34u\right]\Big|_0^x = \frac{1}{128}(-5x^3 - 9x^2 + 102x)$. Thus

$$F_X(x) = \begin{cases} 0 & : x \leq 0, \\ \frac{1}{128}(-5x^3 - 9x^2 + 102x) & : x \in (0,2), \\ 1 & : x \geq 2. \end{cases}$$

(d) (5 points) Find what quantile correspond to $x = 1$.

Solution: The quantile corresponding to $x = 1$ is simply

$$F_X(1) = \frac{1}{128}\left(-5 \times 1^3 - 9 \times 1^2 + 102 \times 1\right) = \frac{88}{128} = \mathbf{\frac{11}{16}}.$$

7. Suppose that a probability that a game of baseball lasts N innings is

$$P(N = n) = \begin{cases} a \times \left(\frac{1}{3}\right)^{n-9}, & : n \geq 9 \\ 0 & : \text{otherwise.} \end{cases}$$

(a) (10 points) Find a.

Solution: We are dealing with a geometric series:

$$1 = \sum_{n=9}^{\infty} P(N = n) = \sum_{n=9}^{\infty} a\left(\frac{1}{3}\right)^{n-9} = a\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n} = a \times \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}a,$$

so $a = \mathbf{\frac{2}{3}}$.

(b) (5 points) Find the mode of X.

Solution: Since $\frac{1}{3} < 1$, for all $n = 9, \ldots,$ $P(X = n+1) = \frac{1}{3}P(X = n) < P(X = n)$, so the series is strictly decreasing. Thus the mode is the value with the smallest index, **9**.

(c) (15 points) What is the probability that a game ends in less that 12 innings if it went into extra-innings (i.e., $N > 9$)?

Solution:

$$\begin{aligned} P(N < 12 | N > 9) &= \frac{P(N < 12 \cap N > 9)}{P(N > 9)} = \frac{P(N = 10 \cup N = 11)}{1 - P(N = 9)} \\ &= \frac{P(N = 10) + P(N = 11)}{1 - P(N = 9)} = \frac{\frac{2}{3} \times \left(\frac{1}{3}\right)^{10-9} + \frac{2}{3} \times \left(\frac{1}{3}\right)^{11-2}}{1 - \frac{2}{3} \times \left(\frac{1}{3}\right)^{9-9}} \\ &= \frac{\frac{2}{9} + \frac{2}{27}}{\frac{1}{3}} = \mathbf{\frac{8}{9}}. \end{aligned}$$

8. (20 points, extra credit) A fair die is rolled twice. Compute the probability mass function, the expectation and the variance for the random variable X equal to the minimum value to appear on the two rolls.

Solution: There are 36 outcomes for the possible rolls of two dice. Any $n = 1, \ldots, 6$ can the maximum of two rolls (one of the rolls can be one; then the maximum is the value of the other roll). In order for the maximum of two rolls to be n, either the first roll is n, and the second roll is more than n ($6 - n$ possibilities), the second roll is n, and the first one is more than n ($6 - n$ possibilities), or both rolls are n (1 possibility), total of $13 - 2n$ possibility. Since each of the 36 possible outcomes are equally likely, the probability mass function for X is

$$
\begin{aligned}
p &= \left\{\left(1, \frac{11}{36}\right), \left(2, \frac{9}{36}\right), \left(3, \frac{7}{36}\right), \left(4, \frac{5}{36}\right), \left(5, \frac{3}{36}\right), \left(6, \frac{1}{36}\right)\right\}. \\
E[X] &= \sum_{n=1}^{N} p(n)\, n = \frac{1 \times 11 + 2 \times 9 + 3 \times 7 + 4 \times 5 + 5 \times 3 + 6 \times 1}{36} = \mathbf{\frac{91}{36}} \approx 2.53. \\
\mathrm{Var}[X] &= E\left[X^2\right] - (E[X])^2 \\
&= \frac{1 \times 11 + 4 \times 9 + 9 \times 7 + 16 \times 5 + 25 \times 3 + 36 \times 1}{36} - \frac{91^2}{36^2} = \frac{301}{36} - \frac{91^2}{36^2} \\
&= \mathbf{\frac{2555}{1296}} \approx 1.97.
\end{aligned}
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