Lecture 9: The Normal Distribution

Devore: Section 4.3

Definition

- A continuous RV X is said to have a *normal distribution* with parameters μ and σ^2, $-\infty < \mu < \infty$ and $0 < \sigma^2$, if the pdf of X is

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

  for all $-\infty < x < \infty$.

- The normal distribution is very important as it describes a very wide variety of data. Heights, weights and other physical characteristics of different populations, measurement errors in scientific experiments and many other types of data are readily described by the normal distribution

- Moreover, sums and averages of a large number of non-normal variables can be described as normal under some suitable conditions.
- It is easy to see that $f(x; \mu, \sigma^2) > 0$; a little more difficult to confirm that

$$\int_{-\infty}^{\infty} f(x; \mu, \sigma^2)\, dx = 1$$

- μ is the mean:

$$E(X) = \mu$$

and σ^2 is the variance:

$$V(X) = \sigma^2$$

Standard Normal Distribution

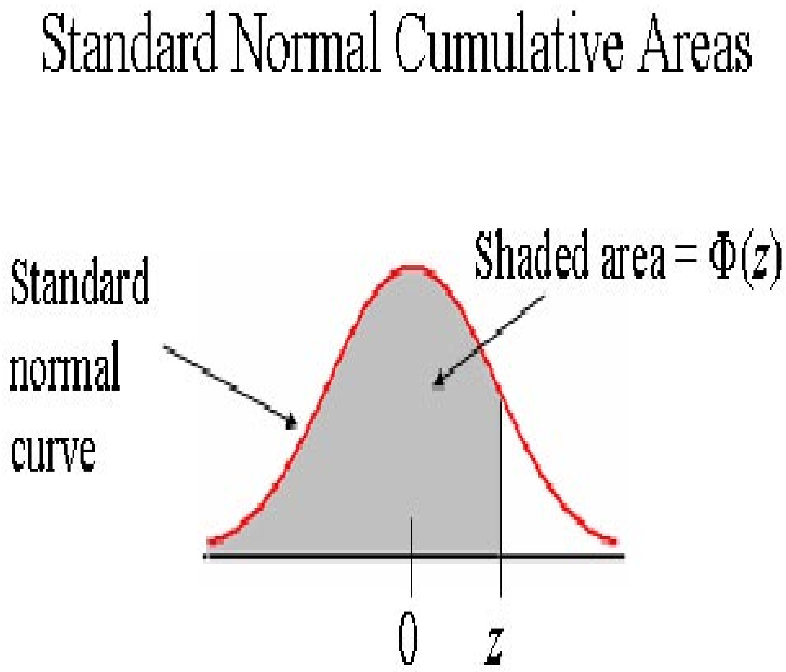
- The normal distribution with parameter values $\mu = 0$ and $\sigma^2 = 1$ is called a *standard normal distribution*.
- A random variable that has a standard normal distribution is called a *standard normal random variable* and is denoted by Z.
- Its pdf is

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

- Its cdf is

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} f(y; 0, 1)\, dy$$

Figure 1:


Standard Normal Cumulative Areas
Standard
normal
curve
Shaded area = Φ(z)
0
z

Examples

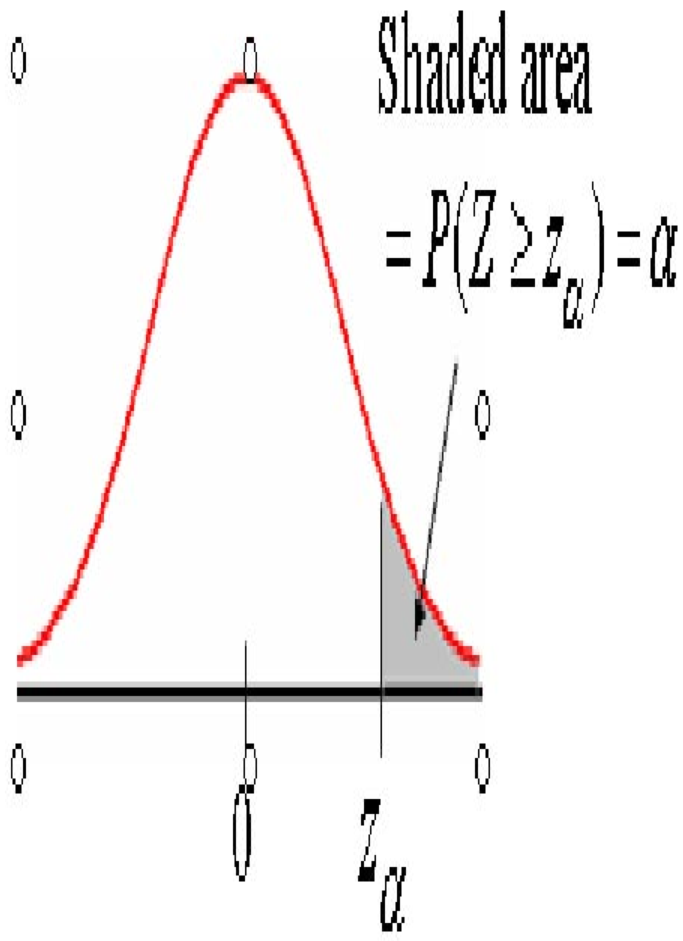
- Let Z be the standard normal random variable. Find

  1. $P(Z \leq 0.85) = 0.8023$ (area under the curve to the left of 0.85)
  2. $P(Z > 1.32) = 1 - P(Z \leq 1.32) = 0.0934$
  3. $P(-2.1 \leq Z \leq 1.78) = P(Z \leq 1.78) - P(Z \leq -2.1) = 0.9625 - 0.0179 = 0.9446$ - the area to the left of 1.78 minus the area to the left of -2.1

Percentiles of the standard normal distribution

- z_α is the value on the measurement axis for which the area under the z curve that lies to the right of it is equal to α

Figure 2:



Example

- **Ex.** Let Z be the standard normal variable. Find z if $P(Z < z) = 0.9278$
  - Look at the table and find an entry $= 0.9278$ then read back to find $z = 1.46$
- Find z such that $P(z < Z < z) = 0.8132$
  - The standard normal distribution is symmetric so $P(-z < Z < z) = 2P(0 < Z < z)$
  - $P(0 < Z < z) = P(Z < z) - \frac{1}{2}$
  - Thus,$2P(Z < z) - 1 = 0.8132$ or $P(Z < z) = 0.9066$
  - From the table, $z = 1.32$

Nonstandard normal distribution

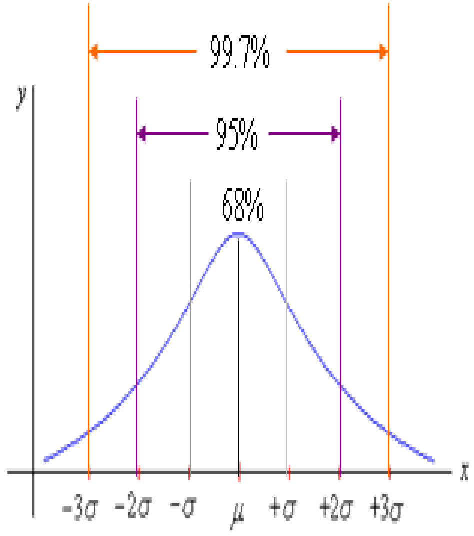
- If X has a normal distribution with mean μ and standard deviation σ, then

$$Z = \frac{X - \mu}{\sigma}$$

has the standard normal distribution

Approximate percentage of area within given standard deviations (empirical rule)

Figure 3:



Example

- Let X be a normal random variable with $\mu = 80$ and $\sigma = 20$
- Find $P(X \leq 65)$

$$P(X \leq 65) = P\left(Z \leq \frac{65-80}{20}\right) = P(Z \leq -.75) = .2266$$

Example

- A particular rash shown up at an elementary school. It has been determined that the length of time that the rash will last is normally distributed with $\mu = 6$ days and $\sigma = 1.5$ days
- Find the probability that for a student selected at random, the rash will last for between 3.75 and 9 days

$$P(3.75 \leq X \leq 9) = P\left(\frac{3.75 - 6}{1.5} \leq Z \leq \frac{9 - 6}{1.5}\right)$$
$$= P(-1.5 \leq Z \leq 2) = 0.9772 - 0.0668 = 0.9104$$

Normal Approximation to the Binomial Distribution

- Let X be a binomial RV based on n trials, each with probability of success p.
- If the binomial probability histogram is not too skewed, X may be approximated by a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$
- Recall that $q = 1 - p$
- **Ex.** At a particular small college the pass rate of Intermediate Algebra is 72%. If 500 students enroll in a semester determine the probability that at most 375 students pass.

- First, $\mu = np = 500 \cdot (.72) = 360$
- Next, $\sigma = \sqrt{npq} = \sqrt{500 \cdot (.72) \cdot (.28)} \approx 10$
- Finally,

$$P(X \leq 375) \approx \Phi\left(\frac{375.5 - 360}{10}\right) = \Phi(1.55) = 0.9394$$