Lecture 8: Continuous Random Variables: an Introduction

Devore: Section 4.1-4.2

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Continuous Random Variables: Definition

- A random variable X is continuous if its set of possible values is an entire interval of numbers (If $A < B$, then any number x between A and B is possible as a value of X).
- Let X be a continuous RV. Then a probability distribution or probability density function (pdf) of X is a function $f(x)$ such that for any two numbers a and b,

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Probability Density Function

- $f(x)$ is a probability density function (pdf) if

  1. $f(x) \geq 0$ for all values of x.
  2. The area of the region between the graph of f and the $x$axis is equal to 1:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Figure 1:

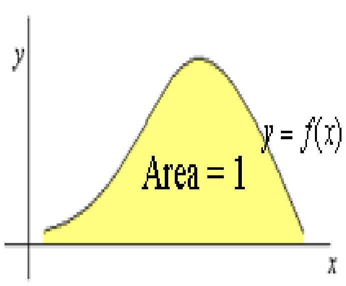
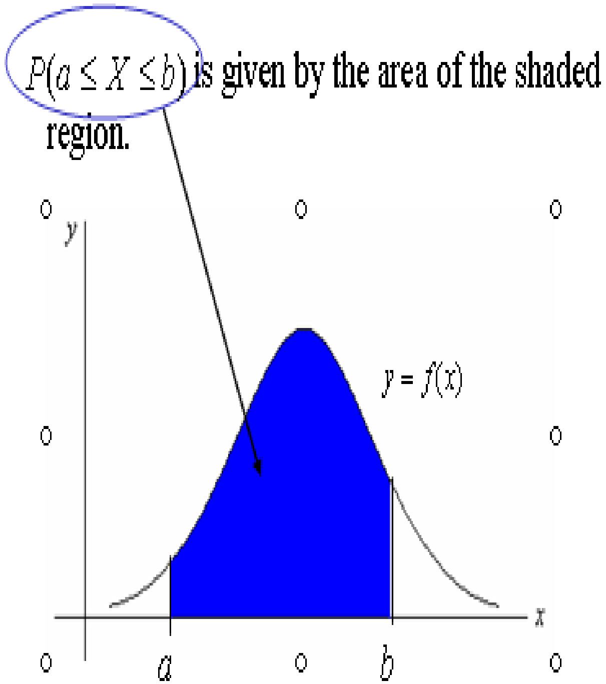


Figure 2:



Uniform distribution

- A continuous RV X is said to have a uniform distribution on the interval $[A, B]$ if the pdf of X is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A} \text{ if } A \leq x \leq B \\ 0 \text{ otherwise} \end{cases}$$

Example

- Consider Ex. 4.3 in Devore The uniform density is

$$f(x) = \begin{cases} \frac{1}{360} \text{ if } 0 \le x \le 360 \\ 0 \text{ otherwise} \end{cases}$$

Figure 3:

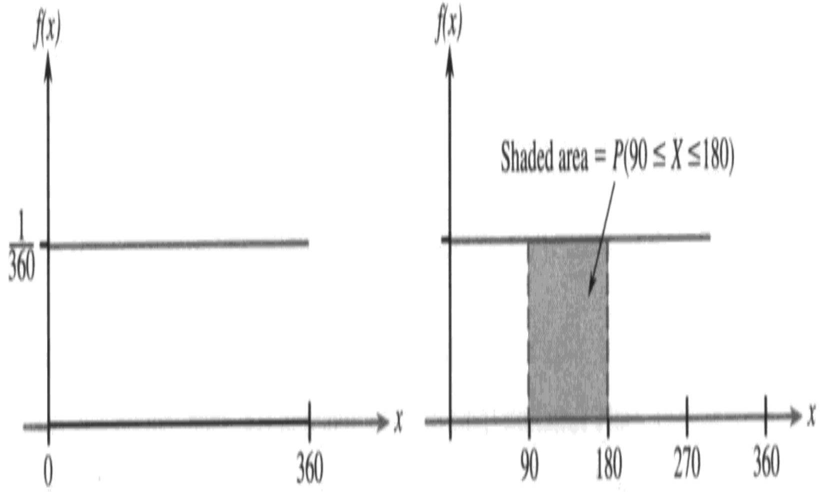


Figure 4.3 The pdf and probability from Example 4.3

- Then,

$$P(90 \leq X \leq 180) = \int_{90}^{180} \frac{1}{360}\, dx = \frac{1}{4}$$

- Note that it will be the same for each of the 4 quadrants of a circular object.

- The probability that the angle is within 90 of the reference line is

$$P(0 \leq X \leq 90) + P(270 \leq X \leq 360) = 2 \cdot \frac{1}{4} = 0.5$$

Probability for a Continuous RV

- If X is a continuous rv, then for any number c, $P(X = c) = 0$.
- For any two numbers a and b with $a < b$,

$$P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$$

Cumulative Distribution Function (cdf)

- The cumulative distribution function $F(x)$ of a continuous RV X is defined for every number x as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$$

- For each x $F(x)$ is the area under the density curve to the left of x.
- **Ex.** Consider Ex. 4.5 in Devore. Note that $F(x) = 0$ if $x < A$ and $F(x) = 1$ if $x > B$
- If $A \leq x \leq B$, we have

$$F(x) = \int_{-\infty}^{x} f(y)\, dy = \frac{x - A}{B - A}.$$

Using $F(x)$ to Compute Probabilities

- Let X be a continuous RV with pdf $f(x)$ and cdf $F(x)$. Then for any number a,

$$P(X > a) = 1 - F(a)$$

- For any numbers a and b such that $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a)$$

Example

- Consider Ex. 4.6 We have the density function

$$f(x; A, B) = \begin{cases} \frac{1}{8} + \frac{3}{8}x \text{ if } 0 \leq x \leq 2 \\ 0 \text{ otherwise} \end{cases}$$

- Then, for any $0 \leq x \leq 2$,

$$F(x) = \int_{-\infty}^{x} f(y)\, dy = \int_{0}^{x} \left(\frac{1}{8} + \frac{3}{8}y \right) dy = \frac{x}{8} + \frac{3}{16}x^2$$

- Based on the above, we have

$$P(1 \leq X \leq 1.5) = F(1.5) - F(1) = \frac{19}{64}$$

Obtaining $f(x)$ from $F(x)$

- If X is a continuous RV with pdf $f(x)$ and cdf $F(x)$, then at every number x for which the derivative $F'(x)$ exists,

$$f(x) = F'(x)$$

- **Ex.** Consider the uniform cdf

$$f(x; A, B) = \begin{cases} 0 \text{ if } x < A \\ \frac{x-A}{B-A} \text{ if } A \leq x \leq B \\ 1 \text{ if } x > B \end{cases}$$

- The pdf is then equal $F'(x) = \frac{1}{B-A}$ for $A \leq x \leq B$ and 0 otherwise

Percentiles

- Let $0 < p < 1$. The $(100p)$ th percentile of the distribution of a continuous RV X is denoted by $\eta(p)$ and is defined from the equation

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)\,dy.$$

- The median of a continuous distribution, denoted by $\tilde{\mu}$ is the 50th percentile. The defining equation is

$$0.5 = F(\tilde{\mu})$$

- That is, half the area under the density curve is to the left of $\tilde{\mu}$.

Expected Value of a RV

- The expected or mean value of a continuous RV X with pdf $f(x)$ is

$$E(X) = \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

- If X is a continuous RV with pdf $f(x)$, then for any function $h(x)$

$$E(h(X)) = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx$$

Variance and Standard Deviation

- The variance of continuous RV X with pdf $f(x)$ and mean μ is

$$V(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx = E(X-\mu)^2$$

- The standard deviation is $\sigma_X = \sqrt{V(X)}$
- The shortcut formula is

$$V(X) = E(X^2) - [E(X)]^2$$

- For any constants a and b,

$$V(aX+b) = a^2 \cdot V(X)$$

and $\sigma_{aX+b} = |a| \cdot \sigma_X$