Expectation existence and **Markov inequality**

- If $\mathbb{E}X^m$ exists for some positive $m$, $\mathbb{E}X^k$ also exists for any $k \leq m$.

- **Markov inequality**: if $u(X)$ is a non-negative function of $X$ s.t. $\mathbb{E}u(X)$ exists, for every positive constant $c$

  \[
P[u(X) \geq c] \leq \frac{\mathbb{E}u(X)}{c}
  \]
Chebyshev inequality

- In Markov inequality, take \( u(X) = (X - \mu)^2 \) where \( \mu = \mathbb{E}X \)
- Select \( c = k^2\sigma^2 \) for a positive \( k \)
- The result follows:

\[
P(\left| X - \mu \right| \geq k\sigma) \leq \frac{1}{k^2}
\]

- In practice, one would select \( k \geq 1 \) to have a meaningful inequality
- Taking \( k\sigma = \varepsilon \) for some \( \varepsilon > 0 \) obtain a commonly used form

\[
P(\left| X - \mu \right| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}
\]
How good Chebyshev inequality is in practice?

- Take $X \sim Unif[-\sqrt{3}, \sqrt{3}]$ with $\mu = 0$ and $\sigma^2 = 1$
- For $k = \frac{3}{2}$ the exact probability is
  \[
P \left( |X| \geq \frac{3}{2} \right) = 1 - \frac{\sqrt{3}}{2}
\]
  while the Chebyshev’s bound is $\frac{1}{k^2} = \frac{4}{9}$
- Take $k = 2$ and the exact probability is $P(|X| \geq 2) = 0$ while
  the Chebyshev bound is $\frac{1}{k^2} = \frac{1}{4}$
- Only the existence of the mean and variance of $X$ is assumed -
  the inequality can be quite conservative!
Example

Let $X$ be the discrete distribution s.t. $p(-1) = \frac{1}{8}$, $p(0) = \frac{3}{4}$ and $p(1) = \frac{1}{8}$.

- Note that $\mu = 0$ and $\sigma^2 = 1$.

- If $k = 2$, the exact probability is $P(|X| \geq 1) = \frac{1}{4}$ - the same as the Chebyshev bound.

- Cannot improve Chebyshev inequality unless extra assumptions about the distribution of $X$ are made.
A much sharper large-deviation inequality is the so-called Chernoff-Bernstein ineq.

Let the random variable $X$ have the mgf $\psi(t) < \infty$ for $t < t_0$ for some $0 < t_0 < \infty$

Let $\kappa(t) = \log \psi(t)$ be the cumulant generating function

Define the rate function of $X$ $I(x) = \sup_{0 < t < t_0} [tx - \kappa(t)]$

Easy transformation results in

$$P(X \geq x) \leq \exp (-I(x))$$
Example

- Take $X \sim N(0, 1)$
- By Chebyshev’s inequality,

\[
P(X \geq x) = \frac{1}{2} P(|X| \geq x) \leq \frac{1}{2x^2}
\]

- By Chernoff-Bernstein inequality, we have

\[
P(X \geq x) \leq \exp \left( -\frac{x^2}{2} \right)
\]
Convex functions

- A function $\phi$ on any real interval is **convex** if for any $x, y$ and for any $0 < \gamma < 1$
  
  $$
  \phi[\gamma x + (1 - \gamma)x] \leq \gamma \phi(x) + (1 - \gamma)\phi(y)
  $$

- If the above inequality is strict, the function $\phi$ is **strictly** convex

- A function $\phi$ is **concave** if $-\phi$ is convex

- **Strict concavity** is defined in the same way as strict convexity
Jensen’s inequality

- For a convex function $\phi$ on an open interval $I$ and $X$ whose support is contained in $I$ s.t. $\mathbb{E}X < \infty$

  $$\phi(\mathbb{E}X) \leq \mathbb{E}[\phi(X)]$$

- If $\phi$ is strictly convex the inequality is also strict unless $X$ is a constant

- The inequality direction is reversed for a concave function
Let $X$ have a discrete uniform distribution: $P(x) = \frac{1}{n}$ for each of $a_1, \ldots, a_n$ where $a_i > 0$

$-\log x$ is a convex function so

$$-\log(EX) \leq E(-\log X) = -\frac{1}{n} \sum_{i=1}^{n} a_i = -\log \left( \prod_{i=1}^{n} a_i \right)^{1/n}$$

Conclude that

$$(a_1 \cdots a_n)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$$
Example

- In the previous result, replace $a_i$ by $\frac{1}{a_i}$

- Confirm that

$$\frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_i}} \leq (a_1 a_2 \cdots a_n)^{1/n}$$

- Relationship between the harmonic mean (HM), geometric mean (GM) and the arithmetic mean (AM):

$$HM \leq GM \leq AM$$