Generating Function (GF)

- For a discrete nonnegative integer-valued random variable $X$ define $G(s) = G_X(s) = \mathbb{E}(s^X) = \sum_{x=0}^{\infty} s^x P(X = x)$

- $G(s)$ is always finite when $s \leq 1$ for any random variable $X$; it may be finite in other regions as well

- Property 1: Let $G(s)$ be finite in some open interval around the origin. Then, $P(X = k) = \frac{G^{(k)}(0)}{k!}$ for any $k \geq 0$

- Property 2: If $\lim_{s \to 1} G^{(k)}(s)$ is finite, the $k$th factorial moment

$$\mathbb{E}[(X(X-1)\ldots(X-k+1)] = G^{(k)}(1) = \lim_{s \to 1} G^{(k)}(s)$$
GF of a sum of independent RV’s; distribution determining property

▶ Property 1: $X_1, \ldots, X_n$ are independent RV’s with GF’s $G_1(s) \ldots, G_n(s)$. Then,

$$G_{X_1 + \ldots + X_n}(s) = \prod_{i=1}^{n} G_i(s)$$

▶ The proof is trivial

▶ Property 2: If two GF’s $G(s)$ and $H(s)$ coincide in any nonempty open interval, then $X$ and $Y$ have the same distribution

▶ The proof is based on a standard property of the power series
Take $X$ a discrete uniform on $\{1, 2, \ldots, n\}$. Then,

$$G(s) = \mathbb{E}[s^X] = \frac{1}{n} \sum_{x=1}^{n} s^x = \frac{s(s^n - 1)}{n(s - 1)}$$

Differentiating at 1 and applying the L’Hôpital’s rule twice, we find that

$$G’(1) = \frac{n + 1}{2}$$
Example

- Take $X$ with a pmf $p(x) = e^{-1} \frac{1}{x!}$, $x = 0, 1, 2, \ldots$.
- Clearly, $p(x) \geq 0$ and $\sum_{x=0}^{\infty} p(x) = 1$.
- The generating function is

$$G(s) = \mathbb{E} [s^X] = \sum_{x=0}^{\infty} s^x e^{-1} \frac{1}{x!} = e^{s-1}$$

- Take the first derivative and conclude that $\mathbb{E}(X) = G'(1) = 1$.
- We discovered the Poisson distribution...
If $M(t) = \mathbb{E}(\exp tX)$ exists in an open neighborhood of size $h$ around zero, it is called a **moment generating function** or mgf.

- $M(t)$ always exists for $t = 0$; if there exists a radius $h$ s.t. $M(t) < \infty$ for any $t : |t| < h$, many useful properties can be derived.

- Also note that the generating function $G(t) = M(\log t)$ whenever $G(t) < \infty$. 

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**Levine**

**STAT 516: Discrete Random Variables**
Main properties

- If the MGF $M(t)$ is finite in some open interval $|t| < h$, then it is infinitely differentiable in that interval and, for any $k \geq 1$,
  \[ \mathbb{E}X = M^{(k)}(0) \]

- (Distribution determining property) For any two random variables $X$ and $Y$ with existing $M_X(t)$ and $M_Y(t)$ the distributions coincide if and only $M_X(t) = M_Y(t)$ for $t \in (-h, h)$ and $h > 0$

- If $X_1, \ldots, X_n$ are independent random variables, and each $X_i$ has a finite mgf in an open interval around 0, we have
  \[ M_{X_1 + \ldots + X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) \]
Let $X$ have the pmf $P(X = x) = \frac{1}{n}$, $x = 1, 2, \ldots, n$

Its mgf is

$$M(t) = \frac{1}{n} \sum_{k=1}^{n} e^{tk} = \frac{e^{t}(e^{nt} - 1)}{n(e^{t} - 1)}$$

To obtain the first derivative at zero; apply L’Hôpital’s rule twice to find

$$E(X) = \frac{n + 1}{2}$$
Example II

Let $X \sim b(1, p)$. The MGF of $X$ is

$$M(t) = pe^t + (1 - p)$$

Verify that $M'(0) = M''(0) = p$
Thus, $\mathbb{E}X = \mathbb{E}X^2 = p$
Example III

- For a fair spinner with the numbers 1, 2 and 3 on it let $X$ be the number of spins until the first 3 occurs.

- If the spins are independent, we have the geometric distribution

$$p(x) = \frac{1}{3} \left( \frac{2}{3} \right)^{x-1}$$

for $x = 1, 2, 3, \ldots$.

- The mgf of $X$ is

$$M(t) = \mathbb{E}(\exp tX) = \sum_{x=1}^{\infty} \exp(tx) \frac{1}{3} \left( \frac{2}{3} \right)^{x-1} = \frac{1}{3} \left( 1 - \exp(t) \frac{2}{3} \right)^{-1}$$

as long as $\frac{2}{3} \exp(t) < 1$ t.i. $t < \log\left(\frac{3}{2}\right)$. 
Nonexistence of mgf

Define

\[ p(x) = \frac{6}{\pi^2 x^2} \]

for \( x = 1, 2, 3, \ldots \) and 0 otherwise

Easy to see that \( \mathbb{E}(tX) = \sum_{x=1}^{\infty} \frac{6\exp(tx)}{\pi^2 x^2} = +\infty \)

Therefore, this distribution doesn't have an mgf
A simple cure is to define a **characteristic function** as

\[ \phi(t) = \mathbb{E}(itX) \]

- \( \phi(t) \) exists for *all* distributions
- \( \phi(t) \) also defines the distribution uniquely
- For any existing moment of \( X \) we have

\[ i^k \mathbb{E}X^k = \phi^{(k)}(0) \]