Lecture 1: Introduction to Probability. Sample spaces, events, probability axioms

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A random experiment can have several possible outcomes.

The sample space of a random experiment $S$ is the set of all its possible outcomes.

Example: tossing the coin once we have the space $S = \{H, T\}$

Example: tossing the coin twice and recording the outcomes produces $S = \{HH, HT, TH, TT\}$

Example: roll a die with 6 faces and record the outcome. Then, $S = \{1, 2, 3, 4, 5, 6\}$. 
The idea of probability

- For us, **probability** is the idealized quantity the **relative frequency** approaches as the number of experiments grows.
- Other approaches are possible...e.g. treating the probability as a personal (subjective) statement of beliefs.
- The theory does not change.
A set is a collection of objects.

Example: if $\mathbb{R}$ is a set of all real numbers, then 0.5 is an element of $\mathbb{R}$.

Notation: $0.5 \in \mathbb{R}$.

If $C = \{x : 0 \leq x \leq 1\}$, we say that $0.5 \in C$ as well.

A set $C$ is countable if it can be enumerated.
Examples

- Neither $C$ nor $\mathbb{R}$ are countable
- The set $\{1, 2, 3, 4, 5\}$ is countable
- The set of all integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is also countable
Basic Set theory II

- A is a subset of $B$ if for every $x \in A$ we have $x \in B$
- Notation: $A \subset B$
- If $A \subset B$ and $B \subset A$, then $A = B$
- A set $A$ is an **empty set** if it has no elements.
- Notation: $A = \emptyset$
Some Relations from Set Theory

- The **union** of two events $A$ and $B$ is the event consisting of all outcomes that are *either* in $A$ or in $B$ or in *both* events. Notation: $A \cup B$. Reading: $A$ or $B$

- The **intersection** of two events $A$ and $B$ is the event consisting of all outcomes that are in *both* $A$ and $B$. Notation $A \cap B$. It is read "$A$ and $B$".

- The **complement** of an event $A$, denoted by $A'$, is the set of all outcomes in $S$ that are *not* in $A$. 
Some Relations from the Set Theory

- The set $A \triangle B$ consists of all outcomes that are either in $A$ or in $B$ but not in both. Reading: a **symmetric difference** of $A$ and $B$

- The **difference** between two sets $A$ and $B$ or a relative complement of $B$ in $A$ $A \setminus B = A \cap B'$ consists of all outcomes in $A$ that are not in $B$

- The notation $A - B$ is also used but is less preferred because it can also mean the set of all $\{a - b|a \in A, b \in B\}$

- As an example, $A \cup B \cup C$ means at least one of $A$, $B$ or $C$ occurred

- $A \cap B \cap C$ means that all of $A$, $B$ and $C$ occurred

- $A \cap B^c \cap C^c$ means that $A$ occurred but not $B$ or $C$
Examples

- If $C_1 = \varnothing$, $C_1 \cup C_2 = C_2$ for any $C_2$
- For any $C$, $C \cup C = C$
- $C_1 = \{1, 2, 3\}$ and $C_2 = \{2, 3, 4\}$
- $C_1 \cup C_2 = \{1, 2, 3, 4\}$ and $C_1 \cap C_2 = \{2, 3\}$
- Consider rolling a die with the sample space $S = \{1, 2, 3, 4, 5, 6\}$.
- $A' = \{4, 5, 6\}$. 
Some Additional Set Theory Relations

- For a sample space $S$, an element $x \in \bigcup_k A_k$ if and only if there exists $k_0$ such that $x \in A_{k_0}$.
- $x \in \bigcap_k A_k$ if and only if $x \in A_k$ for all $k$.
- Example: if $A_k = \{0, 1, 2, \ldots, k\}$ then $\bigcup_k A_k = \{0, 1, 2, 3, \ldots\}$ and $\bigcap_k A_k = \{0, 1\}$.
- Example: let $C_k = \left\{ x : \frac{1}{k+1} \leq x \leq 1 \right\}$, $k = 1, 2, 3, \ldots$. Then, $\bigcup_{k=1}^\infty C_k = \{x : 0 \leq x \leq 1\}$.
- Example: let $C_k = \left\{ x : 0 < x < \frac{1}{k} \right\}$ for $k = 1, 2, 3 \ldots$. Then, $\bigcap_{k=1}^\infty C_k = \emptyset$. 
Venn Diagrams

A

B

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Some operation laws on sets

- **Distributive laws:**
  \[ B \cap (\bigcup_k A_k) = \bigcup_k (B \cap A_k) \]
  and
  \[ B \cup (\bigcap_k A_k) = \bigcap_k (B \cup A_k) \]

- **De Morgan’s laws:** for any two sets \( A_1 \) and \( A_2 \)
  \[ [A_1 \cap A_2]' = A_1' \cup A_2' \]
  and
  \[ [A_1 \cup A_2]' = A_1' \cap A_2' \]
Set functions are important in probability!

Example: $C$ is a set in a two-dimensional space and $Q(C)$ is its area if $C$ is finite

- For a circle $C = \{(x, y) : x^2 + y^2 \leq 1\}$ $Q(C) = \pi$
- For a set $C = \{(0, 0), (1, 1)\}$ $Q(C) = 0$
- For a rectangular triangle $C = \{(x, y) : 0 \leq x, 0 \leq y, x + y \leq 1\}$ $Q(C) = \frac{1}{2}$
Let 

\[ f(x) = \begin{cases} 
(\frac{1}{2})^x & \text{if } x = 1, 2, 3, \ldots \\
0 & \text{elsewhere} 
\end{cases} \]

For \( C = \{0 \leq x \leq 3\}, \)

\[ Q(C) = \sum_{C} f(x) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 = \frac{7}{8} \]
Integration for set functions

Let \( Q(C) = \int_C \exp(-x) \, dx \)

For \( C = \{ x : 0 \leq x < \infty \} \)

\[
Q(C) = \int_0^\infty \exp(-x) \, dx = 1
\]