STAT 516: Some continuous distributions

Lecture 12: Gamma-related distributions

Prof. Michael Levine

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Definition

- **Gamma function** of one argument is an integral

\[ \Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} \, dy \]

that is defined for any \( \alpha > 0 \) and is positive

- Note that \( \Gamma(1) = 1 \); for any \( \alpha > 1 \), integration by parts suggests

\[ \Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \]

which is a **functional equation**

- For an *integer* \( \alpha > 0 \), this becomes

\[ \Gamma(\alpha) = (\alpha - 1)! \]

with the assumption that 0! = 1
Rescale $y = \frac{x}{\beta}$ for $\beta > 0$

Then, obtain that

$$1 = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x\beta} \, dx$$

Define the pdf of the **two argument gamma distribution** as

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x\beta}$$

for any $x > 0$ and 0 otherwise

$\alpha$ is a **shape** and $\beta$ is a **scale** parameter; the usual notation is $X \sim \Gamma(\alpha, \beta)$
Poisson process and gamma distribution

Let $W$ be the random variable describing the length of time needed to obtain $k$ events.

Its cdf is $G(w) = P(W \leq w) = 1 - P(W > w)$.

Clearly,

$$P(W > w) = \sum_{k=0}^{x-1} \frac{(\lambda w)^x e^{-\lambda w}}{x!}$$

The last sum is equal to $\int_\infty^{\lambda w} \frac{z^{k-1}e^{-z}}{(k-1)!} dz$ - just use integration by parts; therefore,

$$G(w) = \int_0^w \frac{\lambda^k y^{k-1}e^{-\lambda y}}{\Gamma(k)} dy$$

for any $w > 0$.

Recognize that $W \sim \Gamma\left(k, \frac{1}{\lambda}\right)$.
Verify that

\[ M(t) = \frac{1}{1 - \beta t}^\alpha \]

Thus, the mean \( \mu = \alpha \beta \) and the variance

\[ \sigma^2 = M''(0) - \mu^2 = \alpha \beta^2 \]

Use R commands `pgamma(x, shape=a, scale=b)` to obtain

\( P(X \leq x) \) and `dgamma(x, shape=a, scale=b)` to get the value

of \( f(x; \alpha, \beta) \)
Beta Distribution

- For two independent $X_i \sim \text{Gamma}(\alpha, \beta)$, $i = 1, 2$, find the distribution of the ratio

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

- Convenient transformation: $y_1 = u(x_1, x_2) = x_1 + x_2$ and $y_2 = u(x_1, x_2) = \frac{x_1}{x_1 + x_2}$

- Verify that this is a one-to-one transformation, $x_1 = y_1 y_2$, $x_2 = y_1 (1 - y_2)$ and the Jacobian $J = -y_1 \neq 0$

- Confirm that the marginal distribution of $Y_2$ is

$$g(y_2) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} y_2^{\alpha + \beta - 1} (1 - y_2)^{\beta - 1}$$

for any $0 < y_2 < 1$ and $0$ elsewhere

- This is **beta distribution** with parameters $\alpha$ and $\beta$
Some thoughts on the beta distribution

- Beta distribution is a generalization of the uniform distribution; it is exactly the uniform distribution when $\alpha = \beta = 1$
- The normalizing constant is
  \[ B(\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} = \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1} \, dx \]
  the value of the **beta function**
- The CDF of the beta distribution is
  \[ F(x) = \frac{B_x(\alpha, \beta)}{B(\alpha, \beta)} \]
  where $B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1 - t)^{\beta-1} \, dt$ is an **incomplete beta function**
- The mean and variance of the beta distribution are $\frac{\alpha}{\alpha + \beta}$, and $\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$, respectively
Example

- Suppose a standardized one hour exam takes 45 minutes on average to finish and the standard deviation of the finishing times is ten minutes. We want to know what percentage of examinees finish in less than 40 minutes.
- If we assume beta distribution, we can obtain values of $\alpha$ and $\beta$ if we know the mean and variance; the results are $\alpha = 4.32$ and $\beta = 1.44$
- The computation gives the answer

\[
P \left( X < \frac{2}{3} \right) = .281
\]