STAT 517: Sufficiency
Measures of Quality of Estimators

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Minimum variance unbiased estimators (MVUE)

- So far, we considered consistency and unbiasedness
- Recall that MLE’s are not always unbiased although they tend to be asymptotically unbiased under a set of regularity conditions
- The model: \( X_1, \ldots, X_n \sim f(x; \theta) \) for \( \theta \in \Omega \)
- For a given \( n > 0 \), \( Y = u(X_1, \ldots, X_n) \) is a minimum variance unbiased estimator (MVUE) of \( \theta \) if \( Y \) is unbiased and the variance of \( Y \) is less than or equal to the variance of every other unbiased estimator of \( \theta \)
Example

Let $X_1, \ldots, X_n \sim N(\theta, \sigma^2)$

Since $\bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)$, $\bar{X}$ is an unbiased estimator of $\theta$...but so is $X_1$.

Clearly, $\text{Var} \bar{X} < \text{Var} X_1$ for any $n > 1$...but $\bar{X}$ is not a minimum variance unbiased estimator!!
Any function $\delta(Y)$ is a **decision function** or a **decision rule**.

A specific values $\delta(y)$ is a **decision**.

To measure how different $\delta(y)$ is from $\theta$, use the **loss function** $L(\theta, \delta(y))$.

The loss function is random...better to use the **risk function**

$$R(\theta, \delta) = \mathbb{E} \{ L(\theta, \delta(y)) \} = \int_{-\infty}^{\infty} L(\theta, \delta(y)) f_Y(y; \theta) \, dy$$

Problem: uniform minimization of risk function over all possible $\theta$ may be impossible.
Choose the mean squared error loss $L(\theta, \delta(y)) = [\theta - \delta(y)]^2$.

How to choose between $\delta(y) = y$ and $\delta_2(y) = 0$? The risk functions are

$$R(\theta, \delta_1) = \mathbb{E} [(\theta - Y)^2] = \frac{1}{n}$$

and

$$R(\theta, \delta_1) = \mathbb{E} [(\theta - 0)^2] = \theta^2$$
Example

- If $\theta = 0$, then the 2nd choice is better...but if $\theta$ is far from zero, $\hat{\delta}_2 = 0$ is a bad choice!
- If considering only $\delta(y)$: $E[(\delta(Y))] = 0$, then $\delta_2$ is not allowed
- Under the latter restriction, we are looking for an MVUE which is, actually, $\bar{X}$ (to be shown later)
- Yet another possible alternative is to use the **minimax criterion**: $\delta_0(y)$ is a minimax decision function if, for all $\theta \in \Theta$,

$$
\max_{\theta} R[\theta, \delta_0(y)] \leq \max_{\theta} R[\theta, \delta(y)]
$$

for any other decision function $\delta(y)$
- Note $R(\theta, \delta_2) = \theta^2$ which is unbounded if $-\infty < \theta < \infty$ and so has to be excluded according to the minimax criterion
- Actually (not proven by us here..) $\delta_1$ is the best choice according to the minimax decision function
Another possibility is simply to define $\delta(X_1, \ldots, X_n)$ without using a statistic $Y$ - will not do it here.

Besides the squared error loss function, can also consider e.g. the **absolute-error loss function**

$$L(\theta, \delta) = \begin{cases} 0 & |\theta - \delta| \leq a \\ b & |\theta - \delta| > a \end{cases}$$

for some $a, b > 0$

This is sometimes called the **goalpost loss function**
A scientist $A$ observes 10 independent trials with prob. of success $0 < \theta < 1$ and has only 1 success

A scientist $B$ observes all trials until the first success which happens to be the 10th

First model: $Y \sim B(10, \theta)$; the second model is $g(z) = (1 - \theta)^{z-1}\theta$ with $z = 10$

A sensible estimate in both cases is the relative frequency: $\hat{\theta} = \frac{y}{n} = \frac{1}{z} = \frac{1}{10}$

$\hat{\theta}$ is unbiased in the first case but not in the second!
The first likelihood is

\[ L_1(\theta) = \binom{10}{y} \theta^y (1 - \theta)^{10-y} \]

The second likelihood is

\[ L_2(\theta) = (1 - \theta)^{z-1} \theta \]

When \( z = 10 \), both are proportional to \((1 - \theta)^9 \theta\)...and both give the same answer \( \hat{\theta} = \frac{1}{10} \)

To a true believer in the likelihood principle the fact that one of them is unbiased does not matter!