Definition

- Expectation of a function $U(X, Y)$: if $U(X, Y) = X$ we have $\mathbb{E}U(X, Y) = \mathbb{E}X$

- New: if $U(X, Y) = (X - \mu_X)(Y - \mu_Y)$ we have the covariance

\[
\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \mathbb{E}(XY) - \mu_X\mu_Y
\]

- The normalized version is the correlation coefficient

\[
\rho = \frac{\text{Cov}(X, Y)}{\sigma_1\sigma_2}
\]
Example: Uniform in a triangle

Let \( f(x, y) = 2 \) if \( x, y \geq 0 \) and \( x + y \leq 1 \)

Recall that \( f_1(x) = 2(1 - x) \) for \( 0 \leq x \leq 1 \) implying that
\[
E(X) = E(Y) = \frac{1}{3}
\]

The covariance is
\[
\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 2 \int_0^1 \int_0^{1-y} xy \, dx \, dy - \frac{1}{9}
\]
\[
= \int_0^1 y(1 - y)^2 \, dy - \frac{1}{9} = \frac{1}{12} - \frac{1}{9} = -\frac{1}{36}
\]

The correlation is
\[
\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = -\frac{1}{\frac{1}{18}} = -\frac{1}{2}
\]
Conditional expectation and regression

- Let \((X, Y)\) with finite and positive marginal variances \(\sigma_1^2\) and \(\sigma_2^2\)
- If \(\mathbb{E}(Y|X)\) is linear in \(X\)
  \[
  \mathbb{E}(Y|X) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(X - \mu_1)
  \]
- Moreover,
  \[
  \mathbb{E}(\text{Var}(Y|X)) = \sigma_2^2(1 - \rho^2)
  \]
Since $\text{Var}(Y|X) = \sigma^2(1 - \rho^2) \geq 0$ we have $-1 \leq \rho \leq 1$

$-1 \leq \rho \leq 1$ ALWAYS - not only when $E(Y|X)$ is linear in $X$ due to Cauchy inequality

Now let $\text{Var}(Y|X) \equiv k$ for all $X$; then, if $\rho = 0$ we have $\text{Var}(Y|X) = \sigma^2$ - the marginal variance

On the other hand, if $\rho^2 \approx 1$ the conditional variance is very small and the distribution is highly concentrated around the conditional mean
Example: uniform distribution on a rectangle

- The joint pdf of $X$ and $Y$ is $f(x, y) = \frac{1}{4ah}$ for $-a + bx < y < a + bx$, $-h < x < h$ and 0 elsewhere.
- The marginal density $f_1(x) = \frac{1}{2h}$ if $-h < x < h$ and 0 elsewhere - uniform with $\sigma_1^2 = \frac{h^2}{3}$.
- The conditional mean is $\mathbb{E}(Y|x) = bx$ and $\text{Var}(Y|x) = \frac{a^2}{3}$.
- Implies that $b = \rho \frac{\sigma_2}{\sigma_1}$ and $\frac{a^2}{3} = \sigma_2^2(1 - \rho^2)$.
- The result:
  \[
  \rho = \frac{bh}{\sqrt{a^2 + b^2 h^2}}
  \]
- Analyze how $\rho$ behaves depending on $a$, $h$ and $b$. 
Let $X_1$ and $X_2$ have the joint pdf $f(x_1, x_2)$ and the marginal pdfs $f_1(x_1)$ and $f_2(x_2)$.

$X_1$ and $X_2$ are independent iff

$$f(x_1, x_2) = f_1(x_1)f_2(x_2)$$

$f(x_1, x_2)$ must be positive on, and only on, the product space $S = S_1 \times S_2$.

The definition is working up to the set $A$ s.t. $P(A) = 0$. 

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Example

- Take the joint pdf \( f(x_1, x_2) = x_1 + x_2 \) when \( 0 < x_1 < 1 \) and \( 0 < x_2 < 1 \) while zero otherwise.
- Verify that
  1. \( f(x_1, x_2) \) is a valid pdf
  2. \( f_1(x_i) = \frac{1}{2} + x_i \) when \( i = 1, 2 \)
- \( X_1 \) and \( X_2 \) are not independent
A simple tool for checking independence

- $X_1$ and $X_2$ are independent iff $f(x_1, x_2) \equiv g(x_1)h(x_2)$
- here $g(x_1), h(x_2)$ are positive functions on $S_1$ and $S_2$, respectively
- The previous example clearly does not satisfy this condition
CDF-based independence criterion

- $X_1$ and $X_2$ are independent iff

$$F(x_1, x_2) = F_1(x_1)F_2(x_2)$$

- Direct consequence is that for any $a < b$ and $c < d$, $X_1$ and $X_2$ are independent iff

$$P(a < X_1 \leq b, c < X_2 \leq d) = P(a < X_1 \leq b)P(c < X_2 \leq d)$$

- Consider the previous example with $a = c = 0$, $b = d = \frac{1}{2}$ to confirm that this condition does not hold.
If $\mathbb{E} u(X_1)$ and $\mathbb{E} v(X_2)$ exist, for independent $X_1$ and $X_2$

$$\mathbb{E} [u(X_1)v(X_2)] = \mathbb{E} u(X_1)\mathbb{E} v(X_2)$$

In particular, if both $u$ and $v$ are identity functions, $\mathbb{E} [X_1X_2] = \mathbb{E} X_1\mathbb{E} X_2$

Note that this result that for independent $X, Y$ $\rho_{X,Y} = 0$
MGFs and independence

- $X_1, X_2$ are independent iff

$$M(t_1, t_2) = M(t_1, 0)M(0, t_2)$$

- Let $X, Y$ have the joint pdf $f(x, y) = e^{-y}$ for $0 < x < y < \infty$ elsewhere.

- The mgf is

$$M(t_1, t_2) = \frac{1}{(1 - t_1 - t_2)(1 - t_2)}$$

for $t_2 < 1$ and $t_1 + t_2 < 1$

- Note that $M(t_1, t_2) \neq M(t_1, 0)M(0, t_2)$ so $X$ and $Y$ are dependent.