For the joint distribution $p_{X_1,X_2}(x_1, x_2)$ and $x_1 : p_{X_1}(x_1) > 0$ the **conditional pmf**

$$p_{X_2|X_1}(x_2|x_1) = \frac{p_{X_1,X_2}(x_1, x_2)}{p_{X_1}(x_1)}$$

for all $x_2 \in S_{X_2}$

Can be easily verified that $\sum_{x_2} p_{X_2|X_1}(x_2|x_1) = 1$ and it is non-negative - it is a proper density!

The usual abbreviation is $p_{2|1}(x_2|x_1)$
If \( f_{X_1}(x_1) > 0 \), let
\[
f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)}
\]
\( f_{X_2|X_1}(x_2|x_1) \) is the **conditional pdf** of \( X_2 \) given that \( X_1 = x_1 \)

Easy to check that \( f_{X_2|X_1}(x_2|x_1) > 0 \) and
\[
\int_{-\infty}^{\infty} f_{X_2|X_1}(x_2|x_1) \, dx_2 = 1
\]

The usual abbreviation is \( f_{2|1}(x_2|x_1) \)
As usual,

\[ P(a < X_2 < b | X_1 = x_1) \equiv P(a < X_2 < b | x_1) = \int_a^b f_{2|1}(x_2 | x_1) \, dx_2 \]

For a function \( u(X_2) \) (of course, can be \( u(X_2) = X_2 \))

\[ \mathbb{E}\{u(X_2) | x_1\} = \int_{-\infty}^{\infty} u(x_2) f_{2|1}(x_2 | x_1) \, dx_2 \]

which is a function of \( x_1 \)

The conditional variance

\[ \text{Var} \left( X_2 | x_1 \right) = \mathbb{E}\{[X_2 - \mathbb{E}(X_2 | x_1)]^2 | x_1\} \]

Immediately,

\[ \mathbb{E}(X_2 | x_1) = \mathbb{E}(X_2^2 | x_1) - [\mathbb{E}(X_2 | x_1)]^2 \]
Example: Uniform in a triangle

- Let $f(x, y) = 2$ for $x, y \geq 0$ and $x + y \leq 1$
- Verify that $f(x|y) = \frac{1}{1-y}$ if $0 \leq x \leq 1 - y$ and zero otherwise
- Conclusion: given $Y = y$ $X$ is distributed uniformly on $[0, 1 - y]$, $\mathbb{E}(X|y) = \frac{1-y}{2}$ and $\text{Var}(X|y) = \frac{(1-y)^2}{12}$
Example: Uniform distribution in a circle

- Let \( f(x, y) = \frac{1}{\pi} \) for any \( x^2 + y^2 \leq 1 \)
- Using our earlier results, conditional density

\[
f(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{1/\pi}{2\sqrt{1-y^2}/\pi} = \frac{1}{2\sqrt{1-y^2}}
\]

for any \(-\sqrt{1-y^2} \leq x < \sqrt{1-y^2}\)
- \(X|Y\) is a uniform on an interval \(-\sqrt{1-y^2} \leq x < \sqrt{1-y^2}\)
  and \(E(X|Y) = 0\)
- The variance \(Var(X|y) = \frac{(2\sqrt{1-y^2})^2}{12} = \frac{1-y^2}{3}\) - decreases as \(y\)
  gets closer to 1; plot to visualize it better!
Example: a two-stage experiment

- For a positive $X$ with pdf $f(x)$ choose $Y$ between 0 and $x$ given $X = x$
- If you only know $Y$...what is your guess for $x$
- Formally, $Y | X = x \sim Unif[0, x]$; find $E X | Y = y$
- Find

$$f(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f(y|x)f(x)}{f_Y(y)} = \frac{1}{x} \mathbf{1}_{x \geq y} f(x) \int_y^{\infty} \frac{1}{x} f(x) \, dx$$

- $E(X | Y = y) = \int_y^{\infty} xf(x|y) \, dx = \frac{1 - F(y)}{\int_y^{\infty} \frac{1}{x} f(x) \, dx}$

where $F$ is the CDF of $X$
Interpretation of a two-stage experiment

- Choose \( f(x) \) to be the Unif[0,1] density
- \( \mathbb{E}(X|Y = y) = -\frac{1-y}{\log y} \) - this is NOT a uniform distribution!
- Now suppose \( f(x) = \frac{1}{x^2} \) for \( x \geq 1 \) - the marginal expectation does not even exist!
- However, \( \mathbb{E}(X|Y = y) = 2y \) which exists for every \( y \)
The Law of iterated expectation

1. If the $\text{Var } X_2 < \infty$, $\mathbb{E}[\mathbb{E}(X_2 | X_1)] = \mathbb{E}(X_2)$

2. $\text{Var } [\mathbb{E}(X_2 | X_1)] \leq \text{Var } X_2$

The sufficiency interpretation - probably the most important for this result!
How to calculate probabilities using The Law of iterated expectation

- For any event $A$, $P(A) = \mathbb{E}I_A$; denote $X = I_A$
- Then,
  \[
P(A) = \mathbb{E}(X) = \mathbb{E}_Y[\mathbb{E}(X|Y = y)] = \mathbb{E}_Y[P(A|Y = y)]
  \]
- Let $X, Y$ be independent $\text{Unif}[0, 1]$ variables and let $Z = XY$
- Then,
  \[
P(Z \leq z) = P(XY \leq z) = \mathbb{E}[XY \leq z] = \mathbb{E}_Y[\mathbb{E}[XY \leq z|Y = y]] = \\
  = \mathbb{E}_{Y=y}[\mathbb{E}[X \leq \frac{z}{y}|Y = y]] = \mathbb{E}_{Y=y}[\mathbb{E}[X \leq \frac{z}{y}]]
  \]
- because $X$ and $Y$ are independent
Now, we know that
\[ P(Z \leq z) = P(XY \leq z) = \mathbb{E}_Y[P \left( X \leq \frac{z}{y} \right)] \]

\[ P \left( X \leq \frac{z}{y} \right) = \frac{z}{y} \text{ if } z \leq y \text{ and } 1 \text{ if } z > y \]

Conclude that \[ P \left( X \leq \frac{z}{y} \right) = z - z \log z \text{ for } 0 < z \leq 1 \]