1. A box has 40 balls, 16 of which are red. You draw three balls
   (a) with replacement;
   (b) without replacement.

   In each case, calculate the expectation and variance, and find the probability that exactly two of the balls are red.

	With Replacement	Without Replacement
Distribution	$X \sim \text{Bin}(n = 3, p = 0.4)$	$X \sim \text{Hyp}(N = 40, n = 3, p = 0.4)$
Expectation	$E(X) = 3(0.4) = 1.2$	$E(X) = 3(0.4) = 1.2$
Variance	$Var(X) = 3(0.4)(0.6) = 0.72$	$Var(X) = \frac{37}{39}3(0.4)(0.6) = 0.683077$
$P(X = 2)$	$p_X(2) = \binom{3}{2}(0.4)^2(0.6) = 0.288$	$p_X(2) = \frac{\binom{16}{2}\binom{24}{1}}{\binom{40}{3}} = 0.291498$

2. A box has 10000 balls, 4000 of which are red. You draw three balls
   (a) with replacement;
   (b) without replacement.

   In each case, calculate the expectation and variance, and find the probability that exactly two of the balls are red. Calculate both using the *actual* distributions.

	With Replacement	Without Replacement
Distribution	$X \sim \text{Bin}(n = 3, p = 0.4)$	$X \sim \text{Hyp}(N = 1000, n = 3, p = 0.4)$
Expectation	$E(X) = 3(0.4) = 1.2$	$E(X) = 3(0.4) = 1.2$
Variance	$Var(X) = 3(0.4)(0.6) = 0.72$	$Var(X) = \frac{9997}{9999}3(0.4)(0.6) = 0.718559$
$P(X = 2)$	$p_X(2) = \binom{3}{2}(0.4)^2(0.6) = 0.288$	$p_X(2) = \frac{\binom{4000}{2}\binom{6000}{1}}{\binom{10000}{3}} = 0.288014$

3. Is it reasonable to approximate the hypergeometric distribution using the binomial when N is large? What if instead of drawing 3 balls, we were drawing 10 balls? Which calculation is easier?

   It should be fairly clear that the approximation is pretty good here - the probability is the same down to the 4^{th} decimal place, and the variance is also very similar. With $n = 3$, all the calculations aren't too hard (especially with a calculator). With $n = 10$, however, unless you have a very good calculator, it may be impossible to solve for $\binom{10000}{10}$ (the number contains 33 digits). However, we can get a very reasonable approximation by using the binomial distribution - which any calculator should be able to do. The above results should make you feel confident that the binomial results will provide a result that is reasonably close to the true answer.

4. You roll a fair die until you see the third six. What is the probability this happens on the 10^{th} roll? Define the random variable you are working with ($X = \ldots$) and state its distribution along with all relevant parameters.

Random Variable	$X =$ the number of rolls until the third six
Distribution	$X \sim$ Negative Binomial$(r = 3, p = 1/6)$
Probability	$P(X = 10) = \binom{9}{2} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7 = 0.046514$

5. What is the probability of getting a heart flush in a 5-card poker hand?

Note: a heart flush requires getting at least 5 hearts in a l-card hand.

Random Variable	$X =$ the number of hearts in a 5 card hand
Distribution	$X \sim$ Hypergeometric$(N = 52, n = 5, p = 1/4)$
Probability	$P(X = 5) = \frac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}} = 0.000495$

6. What is the probability of getting a heart flush in a 7-card poker hand?

Note: a heart flush requires getting at least 5 hearts in a l-card hand.

Random Variable	$X =$ the number of hearts in a 5 card hand
Distribution	$X \sim$ Hypergeometric$(N = 52, n = 5, p = 1/4)$
Probability	$P(X \geq 5) = p_X(5) + p_X(6) + p_X(7) = 0.007641$

$$p_X(5) = \frac{\binom{13}{5}\binom{39}{2}}{\binom{52}{7}} = 0.007128 \qquad p_X(6) = \frac{\binom{13}{6}\binom{39}{1}}{\binom{52}{7}} = 0.000500$$

$$p_X(7) = \frac{\binom{13}{7}\binom{39}{0}}{\binom{52}{7}} = 0.000013$$

7. *You've been asked to hide prizes around your house for your 3-year-old nephew. His chance of finding any given prize is 0.40. How many do you have to hide to know that he will find at least one with probability 0.99?

Let $X =$ the number of prizes found. Then $X \sim$ Binomial$(n =?, p = 0.40)$. We want to pick n such that $P(X = 0) \leq 1 - 0.99 = 0.01$. Therefore:

$$P(X = 0) = \binom{n}{0}(0.4)^0(0.6)^n = 0.6^n \leq 0.01$$

We can solve this by taking the natural log of both sides:

$$n\ln(0.6) = -0.510826n \leq= \ln(0.01) = -4.60517$$

This means that $n \geq 9.0515$. Therefore, we should hide at least 10 prizes.

8. *On a fishing expedition, suppose that each time you cast, you have a 15% chance of catching a fish.

   (a) How many times should you expect to cast your net before you catch your 3^{rd} fish?

Random Variable	X = the number of casts before you catch your third fish
Distribution	$X \sim$ Negative Binomial$(r = 3, p = 0.15)$
Expecatation	$E(X) = \dfrac{3}{0.15} = 20$

   (b) What is the probability of catching three fish within the first 8 catches?

$$P(X \leq 8) = \sum_{x=3}^{8} p_X(x) = 0.105213$$

$$p_X(3) = \binom{2}{2}(0.15)^3(0.85)^0 = 0.003375 \quad p_X(4) = \binom{3}{2}0.15)^3(0.85)^1 = 0.008606$$

$$p_X(5) = \binom{4}{2}(0.15)^3(0.85)^2 = 0.014531 \quad p_X(6) = \binom{5}{2}0.15)^3(0.85)^3 = 0.020727$$

$$p_X(7) = \binom{6}{2}(0.15)^3(0.85)^4 = 0.026427 \quad p_X(8) = \binom{7}{2}0.15)^3(0.85)^5 = 0.031448$$

   (c) Suppose that every morning during your week (7-day) long camping trip, you need to catch 3 fish in order to make breakfast for your family. What is the probability that you catch those three fish within your first 8 catches on exactly four days of your trip?

Note that in the answer provided, X is defined to be the number of casts before catching the third fish, as defined in part(a).

Random Variable	Y = the number of *days* on which you catch 3 fish
Distribution	$Y \sim$ Binomial$(n = 7, p = P(X \leq 8) = 0.105213)$
Probability	$P(Y = 4) = \binom{7}{4}(0.105213)^4(0.894787)^3 = 0.003073$