Combinatorics

So far, we've come up with a very simple formula for calculating probabilities: ______ Unfortunately, calculating $N(E)$ and $N(\Omega)$ can be quite tricky, because there are often a lot more elements in those sets than we first realize. Combinatorics is the field of mathematics used to count the number of objects in a set. To realize why this is such a tricky area, determine what the elements belong to the sample space of the following random experiments - write down a few of the elements. Is it obvious (without any math) how many elements are in the sample space?

1. Draw one card from a standard deck of playing cards.

2. Roll three dice.

3. Draw a five-card poker hand from a standard deck of playing cards.

4. Randomly arrange seven girls in a line.

5. Guess the answer to 10 multiple choice questions (with choices a,b,c, and d) knowing nothing about the subject matter of the test.

As these example should demonstrate, "counting" isn't always easy. The trick is to recognize the patterns. What patterns can you see?

The most important patterns you should have noticed come down to the following two ideas:

Order

In combinatorics, if you're asked whether or not order matters, the real question is this: "If I change the order of the elements, does this need to be counted separately? For example, lets suppose that the seven girls are in the following order: Amanda, Lisa, Jamie, Dana, Karla, Susan, and Lily. By switching Lisa with Dana, the girls are in a different order. Since we're counting orderings, it's fairly obvious that this needs to be counted separately.

In contrast, consider drawing the 5-card hand 7H-QD-5H-10S-5C. If instead you were to draw 7H-QD-10S-5C-5H, this isn't going to change the hand. All that matters here is what cards you draw, not the order in which you drew them.

Finally, consider rolling three dice. Just as with two dice, even if we don't care which dice actually rolls each number, the number of ways to get each possible outcome (say 4-5-2) is determined by how many ways the individual dice can produce those outcomes. Thus, in combinatorics, order does matter for dice.

Replacement

A lot of the sets we're going to talk about are created by combining elements of simpler sets. For example, we create the set of possible rolls of 3 dice by combining the elements from the set of possible results in one roll. There are two ways that this combining can occur:

Without replacement: In sampling without replacement, elements used earlier in the series of mini-experiments are no longer available for subsequent mini-experiments. The most common example is drawing cards from a deck. This means that the sample from which the cards are drawn from changes every time (52, then 51, etc.), and the same card cannot be included twice. This reduces the number of possible hands that can be created.

With Replacement: In sampling with replacement, elements appearing earlier in the series of mini-experiments are still available for subsequent mini-experiments. This means that cards drawn earlier are replaced back into the deck. We can also use this for dice and coins, where the elements appearing earlier cannot disappear later.

Basic Counting Rule (BCR)

Idea

Suppose we have a random experiment that consists of doing the same thing r times - for example, rolling 6 dice, or flipping a coin 10 times. We can instead treat this as a series of r experiments. If each of these smaller experiments has m possible outcomes, then we can calculate the total number of possible outcomes by:

Order

The BCR is used when order matters. Each different ordering must be counted separately.

Sampling

The BCR is used when you sample with replacement.

Examples

1. Suppose you have 5 dice. Die A has 6 sides with sample space $\{1, 2, \ldots 6\}$. Die B has 10 sides with sample space $\{1, 2, \ldots, 10\}$. Die C has 4 sides, $\{2, 4, 6, 8\}$. Die D has 6 sides, $\{3, 6, 9, 12, 15, 18\}$. Finally, Die E has 4 sides: $\{A, B, C, D\}$.

   (a) Suppose you roll dice A,B,C, and D. How many possible outcomes can you have?

   (b) Suppose you roll die E. The dice corresponding to the three sides showing are then rolled. How many possible outcomes are possible for the sum of these dice?

(c) Let S be the sum of the 4 numeric dice, and let $E = \{s : s \in S, s \text{ even}\}$, be the event that the the sum of the 4 numeric dice is even. What is $N(E)$?

2. Suppose Indiana license plates consist of 2 numeric digits (not including 00) followed by one letter, then three digits that might be letters or numbers. How many different license plates are there?

Permutation Rule

Permutations

A permutation is an ____________________ arrangement of objects.

Examples

- Ordered events. Places in a line, numbers, etc.
- Words. "ten" is not the same as "net"; "three" is different from "there".
- Assignments to specific individuals. Think of an assignment as being a part in a play, (different) baseball caps, chores, etc.

Versions

The permutation rule actually has two different versions, depending on whether duplicate objects exist. Duplicate items are considered to be identical, so that switching two duplicate items does not constitute a different arrangement. For example, switching the two 'e's' in the word "three" does not change the word in any way. All copies of the letter 'e' are considered identical.

No Duplicate Items

Conditions:

Action: Permute. The permutation operator is written two different ways:

Special Case: When $m = r$, then

In Words: Suppose we m different objects, and we are taking out r of these objects, one at a time. The first object we take can be any of the m objects, so we have m choices in the first random experiment. No matter what the first object is, we have $m - 1$ choices for the second object. We can continue this pattern until we have selected all r objects. Although the *choices* depend on which objects have been taken already, the *number* of choices for the i^{th} random experiment is always the same! This means that we can use the BCR. Consequently, the total number of combinations is:

Duplicate Items

Conditions:

Action: Divide the total number of permutations by the ____________________ of the number of repeats:

In words: Start by calculating the total number of permutations assuming all items are different. Usually, this will just be the factorial of the total number of items we have (each repeat counted separately). Next, determine how many times each item is repeated. Divide by the factorial of each of these numbers.

Example: Count the number of ways we can permute the letters in the word “SASSY”.

Examples

1. You have 20 different prizes, and a class of 16 students. How many ways can you give each student one prize?

2. If you pick a random number between 0 and 999,999,
   (a) What is the probability that it contains at least one '5'?

   (b) What is the probability it has at least one '5', or exactly one '4'?

3. Consider the letters that make up the word STATISTICS.
   (a) How many ways can these letters be reordered?

   (b) What is the probability that a permutation taken at random has all three 'T's located together?

   (c) What is the probability that the 'C' comes before the 'A'?

4. The Birthday Problem: What is the probability that no two students in this class share the same birthday?

Combination Rule

Combinations

A combination is an ______________________ arrangement of objects.

Examples

- A Committee
- A group of singers in Choir. (contrast with parts in a play)
- A hand of playing cards.

Conditions

The Combination rule is used when we are taking a set of r objects from a collection of m

different objects ____________________________ them.

Action

The combination operator is written two different ways:

In Words

Start by calculating the number of permutations possible when selecting r objects from a collection of m objects. The idea here is almost identical to that of calculating the number of permutations possible when some objects are repeated. We want eliminate the duplication caused by listing all different permutations. The number of permutations of r objects is

___________, so the formula must be:

Examples

1. The US senate consists of 100 senators, two from each state. A 5 person committee is to be chosen from these senators.
   (a) If there are no restrictions, how many ways can a 5 person committee be chosen?

   (b) Suppose that only one senator from each state is allowed. How many ways can the committee be chosen now?

2. How many ways can a group of 10 children be divided into two teams of five children?

3. The Indiana Powerball lottery consists of drawing 5 numbers from $\{1, 2, \ldots, 55\}$ and one number (the POWERBALL) from the numbers $\{1, 2, \ldots, 42\}$. To win, all 6 numbers must match, but the order of the 5 regular numbers is not important.
   (a) What is the probability of winning the Indiana Powerball?

   (b) What is the probability of matching all 5 regular numbers, but not the Powerball?

Complex Problems

The rules from the previous section are often insufficient by themselves to solve many problems in combinatorics. There are many questions that can be asked that require combining several different rules at once, or using other tricks. While not all-inclusive, here are some general principles to help.

- Try to simplify complex problems wherever possible. Try to see if you can separate out smaller problems that the above rules can be applied to.

- Look for mutually exclusive events. Mutually exclusive events are always ____________.
- Sometimes subsets of the same population are used to create both permutations and combinations. Make sure to determine whether or not the same object can exist in both the permutation and the combination. If they can, how would you combine the smaller permutation problem and the combination problem?

  If not, then what would you do?

- Once you have simplified a complex problem into smaller parts, use the same rules to combine the parts.
- Because the combination rule and permutation rule specifically refer to sampling without replacement, generally the situations to which they apply involve objects being drawn - not actions. If each of group of individuals can do two or more *actions*, usually the basic counting rule is more appropriate.

The following table can also help:

	Ordered	**Unordered**
With Replacement	Basic Counting Rule	X
Without Replacement	Permutation Rule	Combination Rule

1. Suppose you play 5 games of poker. In the first 3 games, you draw a hand of 5 cards from a standard 52 card deck. In the last two games, you draw a hand of 7 cards. How many combinations of hands and cards can you have?

2. Suppose a teacher has a class of 30 students. She'd like to choose a class president, vice president, and a council of 4 children. The same student cannot hold more than one position. How many ways can she fill the positions?

3. A password is to consist of 7 or 8 alphanumeric digits. The same number or letter may not be repeated more than once.

4. A director is casting a play that requires 2 children, 4 men, and 5 women, and a "choir" of 4 men and 4 women. If 42 children, 18 men, and 23 women come to the auditions, and any adult who is not cast for a main part is considered for the choir, how many ways can the director fill all the roles?

5. Using the previous problem, what is the probability that David is part of the play, if David is an adult male?

6. Additional practice: Calculate the probability of getting various poker hands. Poker hands consist of 5 cards from a standard deck of playing cards. Order does not matter. Answers to these problems are available at `http://en.wikipedia.org/wiki/Poker_probability`,but I would strongly recommend trying to work them out yourself, and using the web only to check your work.

   (a) A straight flush consists of 5 cards in the same suit in numerical order:

   $$5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit$$

   The Ace can be at the top or the bottom card, but it cannot “wrap around”. I.e. $Q\heartsuit, K\heartsuit, A\heartsuit, 2\heartsuit, 3\heartsuit$ is *not* a straight flush.

   (b) Four of a kind consists of 4 cards of the same rank. There are no restrictions on the 5^{th} card:

   $$7\heartsuit, 7\spadesuit, 7\diamondsuit, 7\clubsuit, Q\heartsuit$$

   (c) A full house consists of 3 cards of one rank, and 2 cards of a second rank:

   $$7\heartsuit, 7\spadesuit, 7\diamondsuit, 3\heartsuit, 3\clubsuit$$

(d) A flush consists of 5 cards of the same suit. Generally, a straight flush is not counted as a flush.

$$8\spadesuit, A\spadesuit, Q\spadesuit, 9\spadesuit, 4\spadesuit$$

(e) A straight consists of 5 cards in numerical order. As with a straight flush, Aces cannot wrap around. A straight flush is generally not counted as a straight.

$$8\clubsuit, 9\heartsuit, 10\diamondsuit, J\heartsuit, Q\clubsuit$$

(f) Three of a kind consists of 3 cards of the same rank. The other two cards cannot have the same rank.

$$7\heartsuit, 7\spadesuit, 7\diamondsuit, Q\heartsuit, 3\clubsuit$$

(g) Two pair consists of 2 cards of one rank, 2 cards of a second rank, and a fifth card that does not match either of the other two ranks.

$$7\heartsuit, 7\spadesuit, 9\diamondsuit, 3\heartsuit, 3\clubsuit$$

(h) A pair consists of 2 cards of one rank. The other three cards cannot have the same rank.

$$A\clubsuit, K\spadesuit, 6\diamondsuit, 3\diamondsuit, 3\clubsuit$$