# Introduction to Probability

## *Think Physics, not Math*

A common mistake made by many students who haven't studied probability or statistics before is to think of it as a math course. Probability is an application of math, like physics, not a particular type of it. We will be using math in this course, but none of it should be new. The hard part of probability is learning when to apply the tools you will learn. Of course, if you do struggle with the math, please come to office hours, and get help!

## *The Purpose of Probability*

Let's continue the comparison between probability and physics for another minute. Physics uses math to model certain types of behavior in the real world. For example, if you were to drop a coin from a specified height, you could calculate the rate at which it dropped using the principles/tools you learn in physics. In physics, the end result is always the same provided the initial conditions are the same (same height, same vacuum, etc.).

Probability allows us to study situations where the end result varies - even when the initial conditions are the same, or at least close enough to the same that we cannot distinguish one from another. Even if we don't know exactly what the outcome will be, probability allows us to make predictions on what the outcome is likely to be. For example:

- If we flip a fair coin 20 times, is it *likely* that it comes up a heads every time?
- If two parents have a daughter, what should they expect the adult height of their child to be? Is the answer likely to change if they have a son? What if the sex of the child is unknown (if, for example, the baby hasn't been born yet)?
- If the stock market has been increasing every day for many days, does that mean that it is likely to increase or decrease tomorrow?
- If children are picked at random for a part in the school play, how likely is it that two best friends are both picked for major roles?

By the end of this course, you should know how probability can be used to find answers to all of these questions, and many others.

## *Course Notation*

Notation is often one of the trickiest, but most important parts of a learning a new subject. Imagine trying to make sense of the formula $A = \pi \times r^2$, the formula for the area of a circle without knowing that $\pi$ is used to represent irrational number $3.141592654\ldots$. If you do not recognize a certain notation, make sure to ask- it's easy to forget new notation a day or two after it's been introduced, and it can be hard to understand the topic of the current day when you're trying to remember what the notation means.

# Review of Sets and Set Notation

*Set*

**Definition:**

**Notation:**

- Sets are generally labeled using ________________ letters, such as __________.

- An element of a set is represented by ________________ letters, such as __________.

- To say that $a$ is an element of the set $A$, we write __________.

- To define a set, we need to describe exactly which elements belong to it. There are several ways to do this, depending on what kinds of elements belong to the set.

  **Non-numeric Sets** Non-numeric sets list each distinct element within ____________________. Objects with multiple copies are only listed once. For example, what is the set of coins made by the US government for circulation?

  When there are a large number of distinct elements, it is sometimes possible to use shortcuts to avoid listing every element. For example, think of a short cut to describe the cards in a standard deck of 52 playing cards?

  **Countable Numeric Sets** The principles from non-numeric sets can always be used on countable numeric sets. However, countable numeric sets often contain patterns that can be used to avoid listing every element separately.

  1. When there are a (small) fixed number of elements in the set, we list all elements in the set between ____________________, as in the non-numeric case. Consequently, set of possible results when a 6-sided die is rolled is:

2. When there is a countably infinite number of numeric elements displaying a clear pattern, we list enough elements at the beginning to establish the pattern, and replace the remaining elements with dots (...). The natural (counting numbers) can be written as:

*Remember that a countably infinite set has infinitely many elements, but in theory, all of them can be listed.*

3. When there is a finite number of numeric elements displaying a clear pattern, in addition to establishing the pattern, it is necessary to show where to stop. Thus, the last element is listed after the dots. Even numbers from 2 to 100 can therefore be written as:

**Uncountable Numeric Sets** Remember that uncountable sets are impossible to list. For example, try to list the numbers from 0 to 1. After 0, no matter how small of a number you choose to come next, you can always find one smaller! Since listing the elements of these sets are impossible, the idea is to instead focus on where the maximum and minimum for a continuous range are. We'll talk about how to work with multiple continuous ranges momentarily.

A ______________________ is used to denote a boundary that includes the boundary point.

A ______________________ is used to denote a boundary that does not include the boundary point, or goes to infinity (or negative infinity).
Examples:

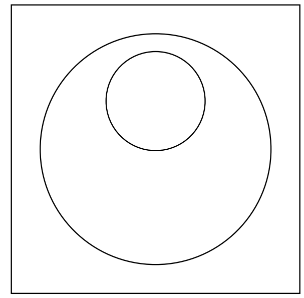
1. The range of numbers from 0 to 1, including 1 but not 0.

2. All non-negative numbers.

## *Special Sets*

There are two special kinds of sets that we need to make note of.

**The empty set:**



**Subset:**

## *Working With Sets*

Consider two sets, $A, B$. There are 3 basic operations we can perform on these sets to create new sets.

- To create a new set that consists of all elements in either set, we take the

- To create a new set that consists of the elements in both sets, we take the

- To create a new set that consists of all elements not in set $A$ (or $B$), we take the

  *Note: Using the compliment requires some knowledge of what the elements not in A are. This should be clear from the context of the problem.*

## *Laws about Sets:*

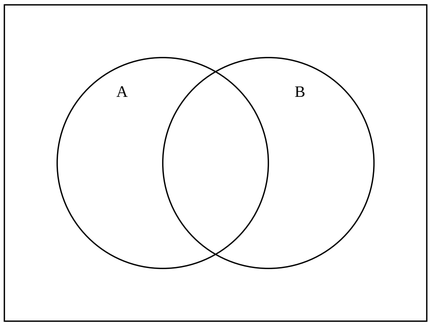
Let $A$,$B$, and $C$ be sets. Then the following laws are always true.

| Commutative Law | Associative Law |
|---|---|
| $A \cap B = B \cap A$<br>$A \cup B = B \cup A$ | $(A \cap B) \cap C = A \cap (B \cap C)$<br>$(A \cup B) \cup C = A \cup (B \cup C)$ |
| **Distributive Law** | **De Morgan's Law** |
| $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | $(A \cup B)^C = A^C \cap B^C$<br>$(A \cap B)^C = A^C \cup B^C$ |

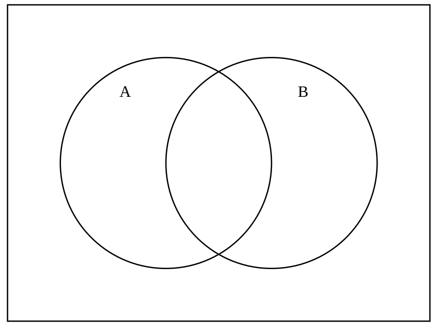
*Visualizing Sets*

Venn Diagrams provide a simple method for visualizing relationships between sets.

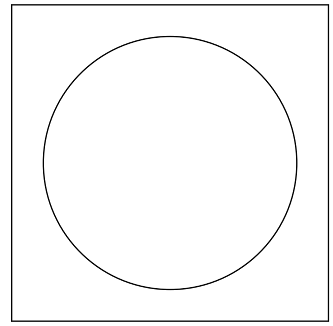
Union: $A \cup B$




Intersection: $A \cap B$




Complement: $A^c$



**Examples**
Let $A = \{1, \ldots, 10\}, B = \{2, 4, \ldots, 20\}$, and $C = \{1, 4, 7, 10, 13, 16, 19\}$. Let $U = \{1, \ldots, 20\}$ be the "entire set", i.e. $A^c = U \cap A^c = \{11, \ldots, 20\}$. Determine the elements of the following sets.

1. $A^c$

2. $A \cup B$

3. $A \cap C$

4. $C \cap (A \cup B)^c$

5. $B \cap (A \cap C)$

6. $(A^c \cup B) \cap C^c$

7. $(A \cap B)^c \cup B$

Draw a Venn Diagram to represent each of the following situations.

1. $A$ is a subset of $B$, and $A$ is a subset of $C$.

2. $A$ is a subset of $B$. $B \cap C \neq \emptyset$. $A \cap C = \emptyset$.

3. $A$ and $B$ are subsets of $C^c$.

# Sets and Probability

## *Random Experiment*

**Definition:**

**Examples:**

## *Sets in Probability*

Sets are used in probability to label groups of possible results of a random experiment. Some important sets used in probability include:

**Sample Space:**

**Event:**

**Outcome:**

**Examples**
Identify the sample space, the desired event, and the outcome.

1. I flip a coin three times looking for at least 2 heads. I actually get a head, then a tail, then a second tail.

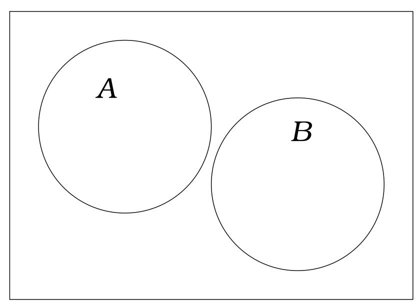
2. I roll a red and a white dice, looking for the sum to be 10. I actually roll a red six, and a white four.

3. I draw a card from a 52 card deck, looking for a face card. I actually get the ace of diamonds.

4. I draw a 5 card poker hand from a 52 card deck, and I'm looking for a flush (all 5 cards are from the same suit.)

## *Mutually Exclusive Events*

**Definition:**




**Note:** Always keep an eye out for mutually exclusive events - they can make many problems much easier.

**Example:** Are the following events mutually exclusive?

1. Getting an even number and getting a number lower than 4 when I roll a 6-sided die.

2. If I roll a die two times, getting an even number on the first roll, and the sum of the two dice being 2.

3. If I roll a die two times, getting an even number on the first roll and getting an odd number on the second roll.

## *Frequentist Definition of Probability*

Define $P(E)$ to be the probability that an event $E$ occurs when a given random experiment is performed. Suppose that this random experiment is repeated $n$ times, where $n$ is a large number. Let $n(E)$ be the number of times that the event $E$ occurs. Then the *frequentist* interpretation of probability states that

***Note****: $P(E)$ is a* function *which takes a set, $E$, and translates into a number.*

## *Axioms of Probability*

There are three basic assumptions about the function $P(E)$ needed for probability to make sense. These are known as the Axioms of Probability, or the ________________________.

**Nonnegativity:**

**Certainty:**

**Additivity:**

## *Calculating Probabilities*

When each element in a sample space is ________________________, then we can calculate the probability of an event by __________________ the probabilities of each element in the event. In other words, let $N(E)$ be the number of elements in the set $E$. Then:

**Example:**
Suppose a random experiment consists of rolling a blue and a red die, and recording the sum of both dice.

1. How many different numbers can the two dice sum to?

2. How many equally likely outcomes are there?

3. What is the probability that the sum of the dice is 5?

4. What is the probability that the sum of the dice is greater than or equal to 9?

5. What is the probability that the sum of the dice is even?

6. What is the probability that the sum of the dice is not 6, 7, or 8?

# Properties of Probabilities

**Probability of the Empty Set:**

**Domination Principle:**

**Complementation Rule:**

***Note***: *The complementation rule is one of the most useful properties in probability.*

**Law of Partitions:**

A **partition** is a group of ____________________ events, $A_1, \ldots, A_n$ in the same sample space where the _______________ of all the events is equal to the sample space:

The **Law of Partitions** states that the probability of an event $B$ can be found by

**General Addition Rule:**

When $A, B$, and $C$ are *not* mutually exclusive, the probability of their union is:

$$P(A \cup B) =$$

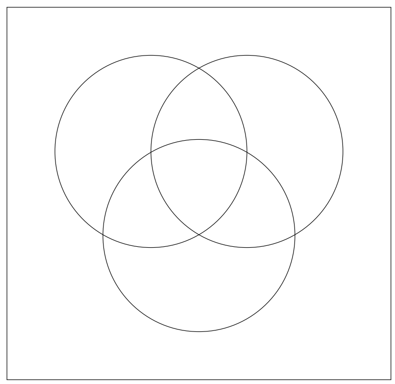
$$P(A \cup B \cup C) =$$

This can be generalized to an arbitrary number of sets using the same principle.

*Examples*

1. A survey is sent to a random sample of 50 undergraduates at Purdue with 3 yes/no questions. Four people responded that they exercise once a week, are from Indiana, and have a family member who has also attended Purdue. Six people exercise once a week and have a family member at Purdue. Seven people are from Indiana and have a family member who has attended Purdue. Seventeen people have a family member who has attended Purdue. Eight people from Indiana responded that they don't exercise, and 8 people who exercise are not from Indiana. Sixteen people responded no to all three questions.

   a. Complete a Venn diagram displaying the number of individuals in each category



   b. If we were to put all the responses into a hat, what is the probability that

      (i) We draw out a response that has a yes for all questions?

      (ii) Draw out a response with exactly two yeses?

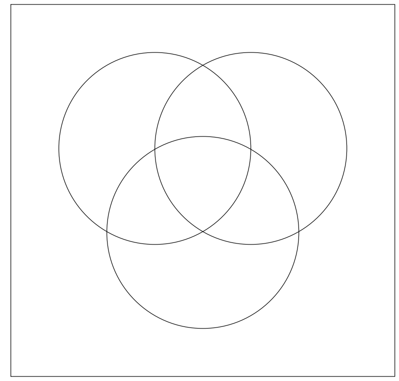
      (iii) Draw out a response with less than two yeses?

      (iv) What is the probability we get a response of someone who is from Indiana, or has a family member who has attended Purdue?

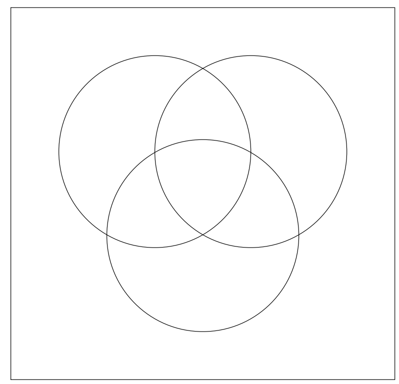
      (v) What is the probability that the respondent answered yes for at least one question?

2. Shade the regions of a 3-way Venn Diagram corresponding to the following regions.

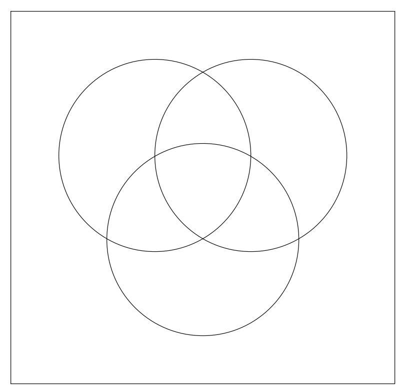
   a. $A \cup B$



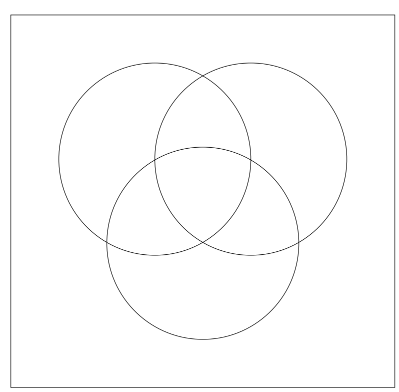
   b. $(A \cup B) \cap C$



   c. $(A \cap B)^c \cap C$



   d. $(C \cup B) \cap (A \cup C^c)^c$



3. If $F$ is the event that a randomly chosen person is female, and $S$ is the event that the person is single, then how would you describe the events that the randomly chosen person is

   a. Married?

   b. A married female?

   c. A single male?

4. In a class with 100 students, 50 like math and 60 like statistics. 15 like both math and statistics. Draw a Venn diagram and answer the following questions.

   a. How many students like only math?

   b. How many like neither math nor statistics?

5. Suppose you roll two fair six-sided dice, one red and one blue.

   a. What is the probability that the two dice will show the same face?

   b. What is the probability that the two dice will show different faces?

   c. What is the probability that the blue dice shows a 3, and the red die a 4?

   d. What is the probability that one of the dice shows a 3, and the other shows a 4?