Part I

PROBABILISTIC STREAMING ALGORITHMS
By 1981, when he first encountered the seeds of what was going to become streaming algorithms at IBM San Jose, Philippe Flajolet had already distilled an impressive set of tools for the analysis of algorithms. But most of these techniques were more or less developed for the problems they were supposed to help solve, and Flajolet was interested in finding completely unrelated problems they could be used to approach.

Probabilistic streaming algorithms, which Nigel Martin and Philippe Flajolet pioneered, proved an exciting such topic. Far from having been a passing interest, Flajolet repeatedly returned to them over more than two decades. His contributions to this subject have been significant but also serve to illustrate a different aspect of his research interests: although these results were eminently mathematical, they showed his understanding of, and appreciation for, implementation level details.

And as this chapter contains a survey by Flajolet himself [24], which already goes a long way exposing the mathematical concepts involved in these algorithms, we have seized the opportunity to approach this topic from a rather more historical perspective.


As a starter, we look at an algorithm Flajolet first wrote about in 1982 [18]. This algorithm is different from the others which will be discussed in this chapter, most notably in that it does not require hash functions. Instead, it is a conceptually simpler introduction to the concept that some theoretical bounds—here the information-theoretical limit that \( \log_2 n \) bits are needed to count up to \( n \)—can be circumvented by making approximations using probabilistic tools.
1.1. Context: spellchecking with no dictionary? The researchers developing Unix at Bell Labs in the mid 70s were fascinated by text processing. Robert Morris wanted to count the occurrences of trigrams in texts—overlapping substrings of three letters. These counts could then be used by typo, a statistic-based spellchecker included in early UNIX distributions, at a time where dictionary-based approaches were out of the question for storage (size and access speed) reasons, see [51] and [47, §3.2].

Unfortunately in this pre-16-bit era, Morris could only fit $26^3$ 8-bit counters into the memory of his PDP-11 mainframe, thus limiting the maximum count to 255: much too small a range to gather any sort of useful trigram count.

Thus instead of maintaining exact counters, Morris suggested making increments in a probabilistic manner. But quickly pointed out that doing so using constant probabilities is not very useful: either the probability of an increment is too large, and the reach is not significantly improved (for example, if you increment every other time, that is with probability $1/2$, then you only allow yourself to count up to 511: you only spare one bit, and the tradeoff is a 50% error); or the probability of an increment is too small, and thus the granularity is too large, in particular making small counts consistently over-estimated (for instance, with a probability of $1/25$, you cannot keep track of values smaller than 25). This approach is also discussed as “direct sampling” by Flajolet at the end of his article.

This suggests the probability of making an increment should not be constant, but instead depend on the current value of the counter. In essence, Morris’ idea [50] is that, with a probability of increment exponential in the value of the counter, it is possible to keep track not of the number $n$ to be counted, but of its logarithm, significantly saving bits (in fact, Morris and his colleagues called the algorithm logarithmic counter).

1.2. Algorithm. The formulation everybody is familiar with, as well as the name Approximate Counting, are due to Flajolet, who, in so doing, contributed greatly to the overall popularity of the algorithm.

Let $N$ be the value we would like to keep track of, i.e., the number of calls to AC-ADDONE; and let $C$ be the value of the (approximate) counter, initially set to 1. If $\text{Ber}(p)$ denotes a Bernoulli random variable (known as a biased coin flip), equal to 1 with probability $p$ and 0 with probability $1 - p$, then adding to and retrieving the value of the counter is done, in its most basic version, with the following procedures:

\begin{align*}
\text{AC-ADDONE}(C) & : & C & \leftarrow C + \text{Ber}(1/2^C) \\
\text{AC-ESTIMATE}(C) & : & \text{return } 2^C - 2
\end{align*}

(1)

to the effect that at all times, an estimate of $N$ is given by $N \approx 2^C - 2$. Indeed when the counter $C$ is equal to 1, the probability of making an increment is 1/2, thus it will take on average 2 calls to AC-ADDONE for the counter $C$ to go from 1 to 2; it then

1. Furthermore, this idea is related to unbounded search in an ordered table, and in recent times has often been presented as such: you are looking for an entry $x$ in an ordered table of unknown and/or infinite size, so you first find out in which geometric interval $[2^k, 2^{k+1})$, $k \geq 0$, $x$ is, then proceed to do dichotomic search in this interval (the way the intervals are subdivided impacts the complexity, see [6]).

2. An overwhelming majority of citations to Morris’ original article date from after 1985, and were usually made in tandem with Flajolet’s paper.
takes 4 calls on average to go from 2 to 3; and more generally, $2^k$ calls to go from $k$ to $k + 1$, to the extent that it requires (on average)

$$2^1 + 2^2 + \ldots + 2^k = \sum_{i=1}^{k} 2^i = 2^{k+1} - 2$$

calls to AC-ADDONE for the counter $C$, initially set to 1, to be equal to $k + 1$.

The accuracy of such a scheme is of roughly one binary order of magnitude—which can be deduced from elementary observations. This accuracy can be improved by changing the base of the logarithm, and making probabilistic increments with probability $q^{-C}$ instead of $2^{-C}$, in which case the estimator then becomes

$$f(C) := \frac{q^C - q}{q - 1}$$

such that the expected value of $f(C)$ after $N$ increments is equal to $N$. The counter will then perform within one $q$-ary order of magnitude; if $q \in (1, 2)$ the accuracy is expected to be improved over the binary version, with a space tradeoff.

While Flajolet greatly clarified Morris’ original algorithms, his other main contribution is to have analyzed them with great finesse. He obtained a more precise characterization of the expected value and of the accuracy involving periodic fluctuations. To this end, he studied a harmonic sum expressing the expected value of $C$ using the Mellin transform discussed in more detail in Chapter 4 of Volume III. It is worthwhile to note that Flajolet was particularly excited to find, first in *Probabilistic Counting*, and then (also through Martin [34] *Approximate Counting*—the analysis of both involving such a complex harmonic sum, or in his words: “I completed the analysis of Approximate Counting and (again!) it has a fairly interesting mathematical structure” (1981). The results provided by Theorem 2 or Section 5, with an expression given as the sum of a linear/logarithmic term, a precise constant term and a trigonometrical polynomial, typically exemplify the sort of fascinating sharp yet easy results yielded by Mellin analysis.

1.3. Recent extensions and applications. Beyond the application introduced as motivation, *Approximate Counting* has been used recurrently in a number of different data compression schemes, where many frequency statistics must be collected, but where their absolute accuracy is not critical (see for instance [34], through which Philippe initially discovered the algorithm, or [47, §3.1]). But although these applications highlight the space-saving aspect of *Approximate Counting*, it would be mistaken to think the algorithm is no longer relevant, with nowadays’ huge storage capacities.

The algorithm has reached great recognition in the streaming literature, as it efficiently computes $F_1$, the first frequency moment (in the terminology of Alon et al. [2]); it is thus often cited for this reason. Beyond that, it has been extended and used in a number of interesting, practical ways. Here are several recent examples.

In 2010, Miklós Csűrös introduced a floating-point version of the counter [13], where accuracy is set in a different way: instead of picking a base $q$ for the logarithmic count, Csűrös suggests splitting the counter into a $d$-bit significant and a binary exponent. The total bits used to count up to $N$ is $d + \log \log N$ bits, but the appreciable
advantage is that small counts, up to $N = 2^d - 1$, are exact\footnote{This variant was developed in the context of mining patterns in the genome \[14\], and coincidentally uses an approach reminiscent of Morris’ original application within the \texttt{typo} program.}. This variant was developed in the context of mining patterns in the genome \[14\], and coincidentally uses an approach reminiscent of Morris’ original application within the \texttt{typo} program.

In 2011, Jacek Cichon and Wojciech Macyna \[11\], in another ingenious application, suggested\footnote{Using \textit{Approximate Counting} in the context of flash memory had already been suggested independently by \[56\], but only as an off-hand comment.} using \textit{Approximate Counting} to maintain counters keeping track of the way data blocks are written in flash memory. Indeed, flash memory is a flexible storage medium, with one important limitation: data blocks can only be written to a relatively small number of times (this can typically be as low as 10 000 times). As a consequence, it is important to spread out data block usage; this is most routinely done by tracking where data has been written through counters stored on a small portion of the flash memory itself. Cichon and Macyna point out that using \textit{Approximate Counting} in this context is pertinent not only because it cuts down on the storage of the counters, but also because the \textit{probabilistic increment} decreases the number of times the counters are actually modified on the flash memory.

This perfectly illustrates the fact that the probabilistic increment at the heart of \textit{Approximate Counting} can be used for two very different reasons: either when \textit{storing} the increments is costly; or when the \textit{action of incrementing itself} is costly. As a parting note, another example: suppose a counter were stored remotely; each increment would require some communication complexity (the size of the message sent remotely to increment the counter); this communication complexity could be considerably decreased, from $O(N)$ to $O(\log N)$, if an \textit{Approximate Counting} type idea were used.

2. An Aside on Hash Functions

With the exception of his paper on \textit{Approximate Counting} which we have just covered, the remainder of Flajolet’s work on probabilistic streaming algorithms uses, at its core, \textit{hash functions}.

2.1. Back in the day. Hash functions (initially also referred to as \textit{scatter storage} techniques) were created in the 1950s for the generic storage/retrieval problem, as an alternate method to, for instance, sorted tables and binary trees \[41, pp. 506-542\]. The abstract premise is that instead of organizing records relative to each other through various schemes of comparisons, the position of a record $x$ in a table is calculated directly by applying a hash function as $h(x)$. As a consequence, with care, hash tables are robust data structures which can have storing/access times that are independent of the number of items stored, and have become extremely popular. In additionally hash functions have found a number of unrelated uses (fingerprinting, dimensionality reduction, etc.).

It is plain to see that the issue here is \textit{collision}, that is when two different elements $x \neq y$ map to the same value $h(x) = h(y)$. At first, hash functions were very specifically designed (as Knuth says, like a “puzzle”) for a particular set of well defined elements, so as to scrupulously avoid any collisions. Predictably that approach
was too unflexible and complex, and soon the goal was only to design hash functions that spread the data throughout the table to attenuate the number of collisions. These properties naturally had to be formalized so algorithms using hash functions could be analyzed.

Thus hash functions began being modelled as associating uniform random variables to each record. At first, this model was very much an idealized approximation. But eventually, as somewhat of an unintended side-effect, hash functions ended up actually becoming good at randomizing data: turning any sort of data into some pseudo-uniform data. Eventually, algorithms began taking advantage of this probabilistic aspect; one particular notable example is Bloom filters, which basically introduced the paradigm of “advocating the acceptance of a computer processing system which produces incorrect results in a small proportion of cases while operating generally much more efficiently than any error-free method.”

Experiment simulations suggested this approach worked surprisingly well, and this usage was cemented in 1977, when Carter and Wegman showed how to build hash function following increasingly stringent probabilistic requirements—including uniformity—therefore providing solid theoretical ground by which to justify the practice.

Yet Carter and Wegman’s “universal hash functions” were rarely used in practice on account of their computational inefficiency, and simpler hash functions yielded surprisingly good results. Quite recently, Mitzenmacher and Vadhan discovered that the reason for this success is that even simple hash functions are very efficient at exploiting the entropy of the data.

### 2.2. From data to uniform variables: reproducible randomness.

Let \( U \) be the possibly infinite set (or universe) of elements that can be hashed; a hash function can be modeled theoretically by a function \( h : U \to \{0, 1\}^\infty \) which is said to uniformize data, when it associates to every element an infinite sequence of random bits, or Bernoulli variables of parameter \( p = 1/2 \), that is

\[
\forall x \in U, \quad h(x) = y_0 y_1 y_2 \cdots \quad \text{such that} \quad \forall k \in \mathbb{N}, \quad P[y_k = 1] = \frac{1}{2}.
\]

(This definition differs from the more traditional one which has hash functions output an integer, but these two definitions are equivalent and related by binary expansion.)

Of crucial importance is the apparent contradiction that the hash functions are, by nature, functions—thus a given hash function \( h \) always associates to an element \( x \) the same value \( h(x) \)—while providing the illusion of randomness. In a strong sense, hash functions provide reproducible randomness, and this concept is at the heart of many probabilistic streaming algorithms.

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5. Another reason why it was important to spread out the load was that linear probing—where upon a collision an element is placed in the closest empty spot—was a popular method to resolve collisions; if elements are clustered together then the next empty spot is much further away from the initial hash index.

6. The name seems to have been coined in a 1976 paper by Severance and Lohman.

This Probabilistic Counting algorithm, as all further ones to be discussed in this introduction, is concerned with efficiently approximating the number of distinct elements (also called cardinality) in a stream, which may of course contain repetitions.

Contrasting with a common, unfortunately lasting, misconception [2], the genesis of Probabilistic Counting was thoroughly practical, to the extent that versions of the algorithm were implemented and in production [4] well before the algorithm was fully analyzed. This makes the contribution unlike most of the literature, essentially theoretical in nature (such as Alon et al. [2] or more recently Kane et al. [38]), since then published on data streaming algorithms.

3.1. Historical context: the birth of relational databases. In the early days of database management systems, at the end of the 60s, accessing data required an intricate knowledge of how it was stored; queries needed to be hard-coded by programmers intimately familiar both with the system and with the structure of the database being queried. As a result, databases were both unwieldy and costly.

Eventually, following the ideas of Edgar Codd at IBM in the 70s [12], there was a large push towards relational databases that could be designed and queried through a high-level language. Obviously, a crucial concern was query optimization—ensuring that the computer-effected compilation of these high-level queries into low-level instructions, produced results within the same order of efficiency as the human-coded access routines of yore. And it soon became apparent the number of distinct elements (in a data column) was the most important statistic on which to base optimization decisions, see [33] or [5, p. 112].

Martin was an IBM engineer in the UK, who worked on one of the first relational databases [55]. When the project came to term in 1978, Martin was granted a sabbatical to conduct original research at the University of Warwick, during which he published works on extendible hashing [46, 10] and data compression [45, 34]. Eventually, he was called to IBM San Jose, to present his unpublished ideas; one of which— influenced by his work on hashing, the emerging ideas on approximating searching and sorting [7] and his prior knowledge of databases—was the original version of Probabilistic Counting.

3.2. Core algorithm. We assume we have a hash function $h$, which transforms every element $y_i$ of the input data stream, into an infinite binary word $h(y_i) \in \{0, 1\}^\infty$, where each bit is independently 0 or 1 with probability 1/2. The algorithm is based on the frequency of apparition of prefixes in these random binary words. Specifically, they were interested in the position of the leftmost 1.

Since each bit is independently 0 or 1 with probability 1/2, we expect that: one in every two words begins with 1; one in every four words begins with 01; one in every $2^k$ word begins with $0^{k-1}1$. Conversely, it is reasonable to assume that in general if

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7. Indeed, Mike Paterson [52] was at Warwick at the time, and a close colleague of Martin.
8. Working with infinite words is a theoretical commodity: of course, in practice, the words are of fixed size—32 or 64 bits usually—and a precise discussion on this is included in the paper. The bottom line is that this in no way limits the algorithm.
we see the prefix $0^{k-1}1$ which occurs with probability $1/2^k$, we can assume there are about $2^k$ words in total.

The algorithm keeps track of the prefix it has seen by maintaining a bitmap or vector of bits, initially all set to 0, bit $i$ is set to 1 when a prefix of length $i + 1$ has been seen. It then make an estimate based on the position of the leftmost zero in this bitmap, which we note $R$.

**Example.** Consider the following stream $S$—in which the infinite words have been truncated to 5 bits, repetitions have been removed, and the prefixes we are interested in have been bolded,

$$ S = 10000, 11101, 00001, 11001, 01100, 10110, 10111, 00111 $$

Once the stream has been processed, the bitmap is equal to $11101$; the position of the leftmost zero (remember positions start in 0) is 3. We can thus make our guess that there are about $2^3$ distinct elements in stream $S$.

Had $S$ contained repetitions, the final value of the bitmap (and consequently our estimate) would have been the same. This is because we are projecting the rank of the leftmost one onto the bitmap—and projections are insensitive to repetitions.

### 3.3. Analysis: no algorithm without math.

So let $R$ be the position of the leftmost zero in the bitmap. Though by construction, it is reasonable to consider that this random variable is on average close to $\log_2 n$, in truth, $R$ has a systematic bias in the sense that there is some $\phi$ such that $E_n[R] \approx \log_2(\phi n)$. As a consequence, if we simply take $2^R$ as an estimate of the number $n$ of distinct elements, then we will be off by non-negligible fraction.

Martin had noticed this, and introduced some ad-hoc correction: look at the three bits following the leftmost zero; depending on their value, adjust $R$ by $\pm 1$. While the reasoning behind this correction was clever, and it does somewhat concentrate the estimates while decreasing the bias, it does not remove it: the estimates produced remain significantly biased.

In essence, this algorithm is in the uncommon position of requiring complex mathematical analysis within its design for its correctness—not just its complexity analysis. This situation would be aptly described by Flajolet’s creed, “no math, no algorithm”; and one of the main results of the paper [28, Theorem 3.A] was to determine that the expected value of the statistic $R$ is

$$ E_n[R] = \log_2(\phi n) + P(\log_2 n) + o(1), $$

where $P$ is an oscillating function of negligible amplitude, so that indeed we may consider $2^R/\phi$ an unbiased estimator of the number of distinct elements.

**Fascinating constants.** Before we move onto how to make this algorithm useful in practice, I wish to make a small digression and discuss this correction constant. The constant $\phi \approx 0.77351 \ldots$ is given exactly by

$$ \phi = 2^{-1/2} e^{-2} \prod_{p=1}^{\infty} \left[ \frac{(4p + 1)(4p + 2)}{(4p)(4p + 3)} \right]^{(-1)^\nu(p)} $$

3. PROBABILISTIC COUNTING (1981-1985)
where $\gamma$ is Euler’s gamma constant and $\nu(p)$ is the number of 1-bits in the binary representation of $p$. Allouch noticed that this constant was related to an unexpected identity due to Shallit [1, §5.2], which provided the starting point for a simplification. Using mainly the identity

\[
\prod_{k=2p}^{2p+1} \left[ \frac{2k+1}{2k} \right] \left( -1 \right)^{\nu(p)} = \left\{ \frac{(4p+1)(4p+2)}{(4p)(4p+3)} \right\} (-1)^{\nu(2p)}
\]

we can obtain the (much slower to converge) expression

\[
\phi = \frac{e^\gamma}{\sqrt{2}} \prod_{p=1}^{\infty} \left[ \frac{2p+1}{2p} \right] (-1)^{\nu(p)}
\]

Some additional details are provided by Steven Finch in his book on mathematical constants [17, §6.8.1].

What is particularly notable is that the elegance and specificity of this constant is the result of Flajolet’s “hands-on” analysis, based on the inclusion-exclusion principle, which is where the number $\nu(p)$ of 1-bits in the binary representation of $p$ comes from. Indeed, the Mellin transform of the probability distribution of $R$ contains the Dirichlet function associated with $\nu(p)$

\[
N(s) = \sum_{k=1}^{\infty} \frac{(-1)^{\nu(k)}}{k^s}.
\]

The product in (6) results from grouping the terms in this Dirichlet function by four. Although the tools Flajolet has developed since would allow for a more straightforward analysis, these would generally not yield such closed-form expressions.

Interestingly, Kirschenhofer, Prodinger and Szpankowski first published in 1992 an alternate analysis of the main estimator [39, 40] which illustrates this well. Instead of using the inclusion-exclusion principle, they frame the analysis of the algorithm in terms of splitting process, which Flajolet had partially written about some years before [29]. Let $R$ be the statistic used by Probabilistic Counting (the leftmost zero in the bitmap) which we have described before, its probability generating function can be described recursively

\[
F_n(u) := \mathbb{E}_n [u^R] \quad \text{and} \quad F_n(u) = \frac{1}{2^n} + u \sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^n} F_k(u).
\]

To obtain this recursion, we consider the bit-vector of all $n$ hashed values, bit after bit, as though they were iterations. On the first iteration, the probability that all first bits are 1 is $1/2^n$, and thus the rank of the leftmost zero in the bitmap will be 0—this contributes $1/2^n$ to the term $u^0$; or else, there is at least one hash value of which the first bit is equal to 0, and thus we make a recursive call with $u$ as multiplicative factor.

Once this functional equation is obtained, the subsequent steps are (now) standard, as we will see: iteration, Poissonization, Mellin. This type of analysis is very similar to that of Adaptive Sampling (see Section 4), and reflects how our angle of
approach has evolved since Flajolet’s initial analysis of *Probabilistic Counting*. The corrective constant which the authors find is

\[
\log_2 \xi = -1 - \frac{1}{(\log 2)^2} \int_0^\infty e^{-x} \prod_{j=0}^\infty \left( 1 - e^{-x2^{j+1}} \right) \frac{\log x}{x} \, dx
\]

and is expected to satisfy \( \xi = \phi \). A direct proof can be derived (as shown by Allouche), and indeed, through numerical integration, we find \( \xi \approx 0.77351 \ldots \) in good agreement with Flajolet’s calculations.

### 3.4. Towards an effective algorithm.

Although the algorithm, at this point, is unbiased, the estimates are typically dispersed by one binary order of magnitude—as expected from the fact that \( R \) can only take integer values.

To improve the accuracy, we could simply run \( m \) simultaneous instances of the algorithm on the same stream, but using a different random hash function for each instance; if we then average these \( m \) estimates, the central limit theorem states this would increase the accuracy by a factor of \( 1/\sqrt{m} \).

This method, however, is not desirable: even assuming we were able to obtain \( m \) good independent uniform hash functions, the computational cost would be huge, especially in light of the fact that so few of hashed values are actually useful.

**Stochastic averaging: making the most out of a single hash function.** The stochastic averaging technique simulates running many concurrent versions of the algorithm using different hash functions, while only using one single hash function—thus at a fraction of the computational cost. As a tradeoff, it delays the asymptotic regime for well-understood reasons, and introduces non-linear distortions.

Instead of running the algorithm in parallel with several hash functions, then taking the average, a very similar effect can be reproduced by splitting the main stream into several substreams. This is done by sampling the first few bits of the hash value to determine in which stream place the value, and discarding these bits. The averaging is called *stochastic* because every instance of an element is distributed to the same substream (instead of just randomly distributing all occurrences in the substreams, which would be useless, as the cardinality of a substream would have no relation with the cardinality of the whole).

One undesirable side-effect of this technique is that the asymptotic regime is significantly delayed, as shown in Figure 1. Indeed while the original algorithm provides comparatively accurate estimates throughout its whole range, we now split the stream into \( m \) substreams—and the quality of the resulting estimates depends intricately on *how many* substreams actually contain elements. It is plain to see that if \( n \ll m \) then the problems are compounded: most substreams will be empty; those that aren’t will only contain a small fraction of the values. As a result, the final average would be significantly worse than what would have been obtained without stochastic averaging.

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9. If the stream has \( N \) total elements, \( n \) of which are distinct, then—per a classical result on records in permutations—only about \( O(\log n) \) of these values are expected to change the state of the bitmap; the rest are just ignored.
Empirical observations suggest that these distortions can be ignored for \( n > 6m \), although recent work shows that for smaller values of \( n \) the distortions can be corrected \cite{44} \S3. The original paper suggested keeping exact counts up to a certain threshold, and then switching to Probabilistic Counting; we will see in next section a different estimation algorithm, Adaptive Sampling, that does not have this issue with small cardinalities, and also how Philippe Flajolet and Marianne Durand found an elegant alternative solution when designing LOGLOG.


In 1984, Mark Wegman—of the universal hash function fame—suggested, in private communications, a new algorithm for cardinality estimation, which he named Sample counting and which avoided the problem of distortions of Probabilistic Counting for small cardinalities.

**Description.** Wegman’s algorithm uses the uniformizing properties of hash functions, as described in Section \[2\], to construct a subset containing a known proportion of all elements in the data stream.

It does so adaptively: it initially assumes all elements in the stream will fit into a cache with \( m \) slots; then as it gets evidence to the contrary (because the cache overflows), it decides to only keep 50% of all objects, and if proven false again then 25%, and so on. And finally, the selection of a subset of elements is done by restricting the hash value of objects that can be in the cache: for instance, if the only elements allowed in the cache are those with hash value prefixed by \( 00 \cdots \), then any element has probability \( 1/4 \) of being in the cache (and thus the cache will contain 25% of all elements, unless it overflows). More formally, the algorithm can be described as in Figure \[2\].
4. ADAPTIVE SAMPLING (1989)

initialize $C := \emptyset$ (cache) and $d := 0$ (depth)

forall $x \in S$ do

\begin{align*}
&\text{if } h(x) = 0^d \cdots \text{ and } x \notin C \text{ then} \\
&\quad C := C \cup \{x\}
\end{align*}

while $|C| > m$ do

\begin{align*}
&d := d + 1 \\
&\quad C := \{x \in C \mid h(x) = 0^d \cdots\}
\end{align*}

return $2^d \cdot |C|$

\textbf{Figure 0.2}. The Adaptive Sampling algorithm.

In the end, the algorithm has a cache $C$ containing any element with probability $1/2^d$; a good statistical guess of the entire number of elements is the $2^d \cdot |C|$. This is what Flajolet proved in his paper [22], along with the accuracy of this estimator.

4.1. The wheels are greased: or how the analysis holds no surprises. In the context of Flajolet’s papers on streaming algorithms, this paper is interesting not for its complexity, but for its simplicity. Indeed, the mathematical structure of the algorithm is, in essence, practically the same as that of Approximate Counting and Probabilistic Counting. But the analysis is here much clearer and simpler—it is only three pages long! This owes to the fact that it is formulated in terms of splitting process [29], and benefits from the progressive refinement and simplification of that type of analysis.

A splitting process simply means that we consider the execution of the algorithm as a branching structure: a tree which contains at its root all elements, and at each node separates the elements which are discarded (in the left subtree) and those that are retained (in the right subtree); this yields a functional divide-and-conquer type equation that has now become easy to solve.

In the same vein, another contemporary article by Greenberg et al. [32], on estimating the number of conflicts in communication channels, bears more than passing resemblance to this algorithm and its analysis.

Other concepts, such as the previously oft-used “exponential approximation”, are now much better understood, routinely used in fact, and no longer justified. In fact, this article marks the first time Flajolet explicitly [22, §3.C] states that the approximation $(1 - a)^n \approx e^{-ax}$ is equivalent to a Poissonization: in the splitting process, instead of considering all possible ways to split $n$ values into two subtrees, Poisson variables of mean $n/2$ are used—which yields a very precise approximation in practice.

4.2. As a sampling algorithm. Despite conceptual strengths, Adaptive Sampling is less accurate than Probabilistic Counting, and though implemented [4] was,

\[\text{(10). Interestingly, as mentioned in Subsection 13.3 this method was also later used by Kirschenhofer et al. [39] to provide a simpler analysis of Probabilistic Counting.}\]

\[\text{(11). It would take several years for the reverse notion to appear: called Depoissonization, it formalizes how to go from the Poisson model to the exact/Bernoulli model.}\]
as far as I know, never used in practice as a cardinality estimation algorithm. But Flajolet quickly realized that it could be used to yield very interesting statistics beyond the number of distinct elements it was initially designed to estimate [23].

Indeed, at any point during the course of its execution, the algorithm (parameterized to use $m$ words of memory) stores a uniform sample of between $m/2$ and $m$ distinct elements taken from the set underlying the stream. Thus elements are sampled independently of their frequency in the stream: an element appearing a thousand times, and another appearing only once would be sampled with equal probability.

Furthermore by attaching frequency counters to the elements, the proportion of various classes of elements can be estimated: for instance, those elements appearing once (called mice, in network analysis) or those appearing more than say, ten times (called elephants), see [42, 43] for detailed analyses.

This algorithm was subsequently rediscovered by several authors, but in particular by Gibbons [30], who most pertinently renamed it Distinct Sampling—which then influenced an algorithm by Bar-Yossef et al. [5, §4].

More recently, the basic idea was popularly generalized as $\ell_p$-sampling, see for instance [49], which samples an element $i \in \{1, \ldots, n\}$, appearing $f_i$ times in the stream, with probability proportional to $f_i^p$ for some specified $p \in \mathbb{R}_>$—in this setting, Distinct Sampling would be related to the special case $p = 0$.

In another direction, Helmi et al. [35] have begun investigating algorithms in the vein of Distinct Sampling, but with the novel feature of being able to control the size of the cache as a function of the number of distinct elements (for instance, you may ask for a uniform sample of $k \log n$ distinct elements).

5. Epilogue

The novel ideas behind these algorithms, and behind Probabilistic Counting in particular, had a lasting impact and contributed to the birth of streaming algorithms. The concepts were further formalized in the groundbreaking paper by Alon et al. [2] in 1996/2000, and from then on, the literature, until then fledgling and rooted in practical considerations, became increasingly expansive and theoretical.

Flajolet’s own further contribution, the LOGLOG family of algorithms, is generally much better known than its predecessors. These algorithms bring small but crucial optimizations: a different statistic that requires a logarithmic-order less memory to track [15]—some algorithmic engineering to avoid extremal values and increase accuracy [13, §5]; and the same gain in accuracy without algorithmic engineering, but through a different averaging scheme involving the harmonic mean [25].

Although these evolutions might seem self-evident now, they also considerably complicate the analysis of the algorithms: the math involved in the analysis of HYPERLOGLOG is severely more complex than that of Probabilistic Counting.

In the 2010s, with the continuing emergence and ubiquity of Big Data, the HYPERLOGLOG algorithm is universally recognized as the most efficient algorithm in practice for cardinality estimation, and it is used by influential companies [36].

12. This idea was first mentioned in the last pages of Flajolet and Martin’s [28] article; but at the time it was not clear the accuracy tradeoff was worth the gain in space—as later highlighted by Alon et al. [3].
Bibliography


[22] Philippe Flajolet. On adaptive sampling. Computing, 43:391–400, 1990. The article has volume 34 printed at the top, but it is actually contained in volume 43. For preliminary version, see [21].


