1. The testing booklet contains 8 questions.

2. Permitted Texas Instruments calculators:
   - BA-35
   - BA II Plus
   - BA II Plus Professional
   - TI-30Xa
   - TI-30X IIS (solar)
   - TI-30X IIB (battery)

   The memory of the calculator should be cleared at the start of the exam, if possible.
   No other calculators are permitted.

3. Simplify the answers and write them in the boxes provided.

4. Some partial credit may be awarded at the discretion of the professor.

5. Show all of your work in the exam booklet.

   Answers without justification may be viewed as exhibiting academic dishonesty.

6. Extra sheets of paper are available from the professor.

   It might be helpful to know the following fact:
   Suppose $X_1, X_2, \ldots, X_n$ are $n$ independent, identically distributed, continuous random variables, each with density $f(x)$ and cumulative distribution function $F(x)$. Let $X_{(j)}$ denote the $j$th smallest of $X_1, \ldots, X_n$. Then $X_{(j)}$ has density function

   $f_{X_{(j)}}(x) = \frac{n!}{(n-j)!(j-1)!}[F(x)]^{j-1}[1 - F(x)]^{n-j}f(x).$
1. Consider two independent, continuous random variables $X$ and $Y$ that are each uniformly distributed on the interval $(0, 10)$. Given that $X + Y \leq 10$, find the conditional probability that $\max(X, Y) \leq 7$.

**Answer.** We note that $\max(X, Y) \leq 7$ if and only if $X \leq 7$ and $Y \leq 7$. The desired probability is

$$P(\max(X, Y) \leq 7 \mid X + Y \leq 10) = \frac{P(X \leq 7 \text{ and } Y \leq 7 \mid X + Y \leq 10)}{P(X + Y \leq 10)}$$

**Method 1.** Viewing the picture, we see that the area where $X \leq 7$ and $Y \leq 7$ and $X + Y \leq 100$ is $50 - \frac{9}{2} - \frac{9}{2} = 41$, and the total area where $X + Y \leq 100$ is 50. Since the joint distribution is uniform, dividing the two areas yields the desired probability, namely, $41/50$.

![Graphical representation](image)

**Method 2.** The probability in the denominator is $\int_0^{10} \int_0^{10-x} \frac{1}{100} \, dy \, dx = 1/2$, and the probability in the numerator is $\int_3^7 \int_0^{10-x} \frac{1}{100} \, dy \, dx = \frac{21}{100} + \frac{1}{3} = \frac{41}{100}$. So the desired probability is $\frac{41/100}{1/2} = 41/50$. 

![Another derivation](image)
2. Consider two random variables \( X \) and \( Y \) with joint probability mass function 

\[
p_{X,Y}(x, y) = P(X = x, Y = y) = (1/2)^x \quad \text{for integers } x, y \text{ with } 2 \leq y \leq x < \infty
\]

and \( p_{X,Y}(x, y) = 0 \) otherwise.

**Question A:** For an integer \( x \geq 2 \), find the conditional probability mass function \( p_{Y|X}(y|x) \) of \( Y \), given \( X = x \).

**Answer.**

The marginal probability mass function of \( X \) is 

\[
p_X(x) = \sum_{y=2}^{x} (1/2)^x = (x-1)(1/2)^x,
\]

so the conditional probability mass function is 

\[
p_{Y|X}(y|x) = \frac{(1/2)^x}{(x-1)(1/2)^x} = \frac{1}{x-1} \quad \text{for } 2 \leq y \leq x \text{ and } p_{Y|X}(y|x) = 0 \text{ otherwise}.
\]

**Question B:** For an integer \( y \geq 2 \), find the conditional probability mass function \( p_{X|Y}(x|y) \) of \( X \), given \( Y = y \).

**Answer.** The marginal probability mass function of \( Y \) is 

\[
p_Y(y) = \sum_{x=y}^{\infty} (1/2)^x = (1/2)^y + (1/2)^{y+1} + (1/2)^{y+2} + \cdots = \frac{(1/2)^y}{1 - 1/2} = (1/2)^{y-1},
\]

so the conditional probability mass function is 

\[
p_{X|Y}(x|y) = \frac{(1/2)^x}{(1/2)^{y-1}} = (1/2)^{x-y+1} \quad \text{for } y \leq x \text{ and } p_{X|Y}(x|y) = 0 \text{ otherwise}.
\]
3. Let $X$ and $Y$ be independent random variables, as follows. Let $X$ be a continuous random variable with uniform distribution on the interval $(0, 3)$. Let $Y$ be a Poisson($\lambda$) random variable. Find $P(X < Y)$.

[You do not need to simplify your answer.]

**Answer.**

**Method 1.** If $Y = 0$, then we cannot have $X < Y$. If $Y = 1$, then $X < Y$ with probability $1/3$. If $Y = 2$, then $X < Y$ with probability $2/3$. If $Y \geq 3$, then definitely $X < Y$. Thus, the desired probability is $(1/3)P(Y = 1) + (2/3)P(Y = 2) + (1)P(Y \geq 3) = (1/3)(e^{-\lambda} \frac{\lambda^1}{1!}) + (2/3)(e^{-\lambda} \frac{\lambda^2}{2!}) + (1) \left( 1 - e^{-\lambda} \frac{\lambda^0}{0!} - e^{-\lambda} \frac{\lambda^1}{1!} - e^{-\lambda} \frac{\lambda^2}{2!} \right)$.

**Method 2.** We can write the derivation above using a direct calculation (this is really the same argument). We have

\[
P(X < Y) = P(X < Y \mid Y = 0)P(Y = 0) + P(X < Y \mid Y = 1)P(Y = 1) + P(X < Y \mid Y = 2)P(Y = 2) + P(X < Y \mid Y \geq 3)P(Y \geq 3)
\]

\[
= (0) \frac{e^{-\lambda} \lambda^0}{0!} + (1/3) \frac{e^{-\lambda} \lambda^1}{1!} + (2/3) \frac{e^{-\lambda} \lambda^2}{2!} + (1) \left( 1 - e^{-\lambda} \frac{\lambda^0}{0!} - e^{-\lambda} \frac{\lambda^1}{1!} - e^{-\lambda} \frac{\lambda^2}{2!} \right)
\]

\[
= 1 - e^{-\lambda} \left( 1 + \frac{2\lambda}{3} + \frac{\lambda^2}{6} \right)
\]
4. In a large population, the height of a randomly chosen individual is approximately normal. The average height is 70 inches. With probability 10.03%, a person’s height is 72 inches or more. [Hint: The standard deviation can easily be calculated with this information.]

A person is “very tall” if the person’s height is 74 inches or more. In a collection of 100 people, approximate the probability that 2 or more individuals are “very tall”.

**Answer.** Let $Y$ denote a random person’s height and $\sigma$ the standard deviation. So
\[
.1003 = P(Y > 72) = P \left( \frac{Y - 70}{\sigma} > \frac{72 - 70}{\sigma} \right) \approx P(Z > 2/\sigma) = 1 - \Phi(2/\sigma), \text{ so } \Phi(2/\sigma) = .8997.
\]
Thus $2/\sigma = 1.28$, so $\sigma = 1.5625$.

The probability that a person is “very tall” is
\[
P(Y > 74) = P \left( \frac{Y - 70}{1.5625} > \frac{74 - 70}{1.5625} \right) \approx P(Z > 2.56) = 1 - \Phi(2.56) = 1 - .9948 = .0052.
\]

Let $X = X_1 + \cdots + X_{100}$, where $X_j = 1$ if the $j$th person is “very tall”, and $X_j = 0$ otherwise. Then the desired probability is $P(X \geq 2)$.

**Method 1.** We can compute directly
\[
P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - (.9948)^{100} - (100)(.9948)^{99}(.0052) = .0959.
\]

**Method 2.** We can use a Poisson approximation to the Binomial, with parameter $\lambda = np = (100)(.0052) = .52$, to obtain
\[
P(X \geq 2) \approx 1 - e^{-52} - .52e^{-52} = .0963.
\]

**Method 3.** We can use a Normal approximation to the Binomial, with mean $np = (100)(.0052) = .52$ and variance $np(1 - p) = (100)(.0052)(.9948) = .5173$, to obtain
\[
P(X \geq 2) = P(X \geq 1.5) = P \left( \frac{X - .52}{\sqrt{.5173}} \geq \frac{1.5 - .52}{\sqrt{.5173}} \right) \approx P(Z \geq 1.36) = 1 - \Phi(1.36) = 1 - .9131 = .0869.
\]
A deck of 52 cards has 4 distinct Aces and 48 other cards (not Aces). The following procedure is used to find all 4 distinct Aces in the deck. Do the following repeatedly:

The deck of 52 cards is thoroughly shuffled. Exactly one card is selected. The selected card is checked, to see whether it is an Ace. In either case (i.e., whether or not it is an Ace), the card is put back into the deck, all 52 cards are thoroughly shuffled, and the procedure begins again.

The procedure stops immediately after all 4 distinct Aces have appeared.

How many cards do we expect to examine during this procedure?

**Answer.** When \( j \) Aces have already appeared, let \( X_j \) denote the number of additional cards needed until the \((j+1)\)st card appears. Then \( X_0, X_1, X_2, X_3 \) are independent Geometric random variables, where \( X_j \) has probability \( \frac{4-j}{52} \) of success (i.e., of a new Ace appearing) for each card selected. Thus \( E[X_j] = \frac{52}{4-j} \). So the total number of cards we expect to examine is \( E[X_0 + X_1 + X_2 + X_3] = E[X_0] + E[X_1] + E[X_2] + E[X_3] = \frac{52}{4} + \frac{52}{3} + \frac{52}{2} + \frac{52}{1} = \frac{325}{3} \approx 108.3 \).
Each smoke detector device has two batteries with independent, exponential lifetimes. The mean lifetime of each battery is 36 months, i.e., each battery is exponential with parameter $\lambda = 1/36$. A smoke detector will stop functioning if either battery dies.

A house has five smoke detector devices. What is the probability that all five smoke detector devices continue to work for at least 1 month?

**Answer.** The time until the first smoke detector dies is $Y = \min(X_1, \ldots, X_{10})$ where $X_j$ is the lifetime of the $j$th battery (five smoke detector devices, two batteries each, so ten batteries altogether).

**Method 1.** The minimum of a collection of independent exponential random variables is also exponential with parameter equal to the sum of the parameters. Thus $Y$ is exponential with parameter $\frac{1}{36} + \cdots + \frac{1}{36} = \frac{10}{36}$.

**Method 2.** Note that $Y$ is the 1st order statistic of $X_1, \ldots, X_{10}$, so $Y$ has density $\frac{10!}{9!} \left[1 - e^{-(1/36)x}\right]^9 e^{-(1/36)x} \frac{1}{36} e^{-(1/36)x} = \frac{10}{36} e^{-(10/36)x}$, and thus $Y$ is exponential with parameter 10/36.

**Conclusion to Method 1 or Method 2.** Either way, we conclude that $P(Y > 1) = e^{-(10/36)(1)} \approx .757$.

**Method 3.** Another method is to recognize that all of the devices will work for at least 1 month if and only if all 10 batteries work for at least 1 month. The probability that a given battery works for at least 1 month is $e^{-1/36}$, so the desired probability is $(e^{-1/36})^{10} = e^{-10/36}$. 
7. A total of $t$ people simultaneously enter an elevator at the basement. Behaving independently of each other, each person randomly chooses one of the floors 1, 2, \ldots, $n$ on which to exit the elevator; all $n$ choices are equally likely. The elevator only stops at a floor if one or more people will exit at that floor.

How many stops do we expect the elevator to make?

**Answer.** The total number of stops that the elevator makes is $X_1 + \cdots + X_n$, where $X_j = 1$ if the elevator stops on the $j$th floor and $X_j = 0$ otherwise. We note that $X_j = 0$ if and only if none of the $t$ people exit at floor $j$, so $P(X_j = 0) = \left(\frac{n-1}{n}\right)^t$. Thus $E[X_j] = P(X_j = 1) = 1 - \left(\frac{n-1}{n}\right)^t$. We conclude that $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] = n \left(1 - \left(\frac{n-1}{n}\right)^t\right)$. 

8. Consider jointly-distributed continuous random variables $X$ and $Y$ with joint density

$$f_{X,Y}(x, y) = \begin{cases} 
6(y - x) & \text{for } 0 \leq x \leq y \leq 1 \\
0 & \text{otherwise.}
\end{cases}$$

Find $P(X < \frac{1}{3} \mid Y = \frac{1}{2})$.

**Answer.** For $0 \leq y \leq 1$, the marginal density of $Y$ is $f_Y(y) = \int_0^y 6(y - x) \, dx = 6 \left( yy - \frac{x^2}{2} \right) \bigg|_{x=0}^{y} = 3y^2$. Thus, the conditional density of $X$ given $Y = y$ is $f_{X|Y}(x \mid y) = \frac{6(y-x)}{3y^2}$.

In particular, for $Y = 1/2$, the conditional density of $X$ is $f_{X|Y}(x \mid \frac{1}{2}) = \frac{6(\frac{1}{2} - x)}{3(1/2)^2} = 8 \left( \frac{1}{2} - x \right)$. So the desired probability is $P(X < \frac{1}{3} \mid Y = \frac{1}{2}) = \int_0^{1/3} 8 \left( \frac{1}{2} - x \right) \, dx = 8 \left( \frac{1}{2} x - \frac{x^2}{2} \right) \bigg|_{x=0}^{1/3} = 8 \left( \frac{1}{6} - \frac{1}{18} \right) = 8/9$. 