1. The testing booklet contains 8 questions.

2. Permitted Texas Instruments calculators:
   - BA-35
   - BA II Plus
   - BA II Plus Professional
   - TI-30Xa
   - TI-30X IIS (solar)
   - TI-30X IIB (battery)
   The memory of the calculator should be cleared at the start of the exam, if possible. No other calculators are permitted.

3. Simplify the answers and write them in the boxes provided.

4. Some partial credit may be awarded at the discretion of the professor.

5. Show all of your work in the exam booklet.
   - Answers without justification may be viewed as exhibiting academic dishonesty.

6. Extra sheets of paper are available from the professor.
1. Let $c$ be a fixed constant with $0 < c < 1$. (We are not solving for $c$; here, $c$ is just fixed.) Let $X$ have cumulative distribution function

$$F(x) = \begin{cases} 
0 & \text{if } x < -1, \\
1 - c & \text{if } -1 \leq x < 0, \\
1 - c + \frac{1}{2}cx & \text{if } 0 \leq x \leq 2, \\
1 & \text{if } x > 2 
\end{cases}$$

Find $P(X \geq 1)$. Also find the values $P(X = -1)$ and $P(X = 0)$.

**Answer.** Method 1. We have $P(X \leq 1) = F(1) = 1 - c + \frac{1}{2}(c)(1) = 1 - \frac{1}{2}c$, so $P(X > 1) = 1 - (1 - \frac{1}{2}c) = \frac{1}{2}c$. Since $F(x)$ is continuous at $x = 1$, it follows that $P(X \geq 1) = \frac{1}{2}c$ too.

We see that $P(X \leq -1) = F(-1) = 1 - c$. On the other hand, $P(X \leq x) = F(x) = 0$ for $x < -1$, so $X$ is never smaller than $-1$. Thus, the mass $1 - c$ must all be located at the point $x = -1$, so $P(X = -1) = 1 - c$.

[Method 2: $P(X = -1) \approx P(X \leq -1) - P(X \leq -1.000001) = (1 - c) - 0 = 1 - c$. The approximation still holds if we replace $-1.000001$ by a value even closer to $-1$.]

We see that $P(X \leq 0) = F(0) = 1 - c + \frac{1}{2}(c)(0) = 1 - c$. On the other hand, also $P(X \leq x) = F(x) = 1 - c$ for $-1 \leq x < 0$, so $F(x)$ is continuous at the point $x = 0$ and therefore has no mass at this point. So $P(X = 0) = 0$.

[Method 2: $P(X = 0) \approx P(X \leq 0) - P(X \leq -0.000001) = (1 - c) - (1 - c) = 0$. The approximation still holds if we replace $-0.000001$ by a value even closer to 0.]

1
2. Let $X$ have cumulative distribution function

$$F(x) = \begin{cases} 
0 & \text{if } x < 0, \\
\frac{1}{5}x & \text{if } 0 \leq x \leq 5, \\
1 & \text{if } x > 5
\end{cases}$$

and let $Y = X^2$. Find $P(Y + 2 \geq 3X)$.

**Answer.** Method 1. We have $P(Y + 2 \geq 3X) = P(X^2 + 2 \geq 3X) = P(X^2 - 3X + 2 \geq 0) = P((X - 2)(X - 1) \geq 0)$. Notice that $(X - 2)(X - 1) \geq 0$ when $X \geq 2$ or $X \leq 1$. These are disjoint possibilities, so we obtain $P(Y + 2 \geq 3X) = P(X \geq 2) + P(X \leq 1)$.

We note that $X$ is uniform on the interval $[0, 5]$ (if this was not clear from looking at $F(x)$ directly, then just note that the density of $X$ is $f(x) = \frac{1}{5}$ for $0 \leq x \leq 5$ and $f(x) = 0$ otherwise).

So $P(X \geq 2) = \frac{3}{5}$ and $P(X \leq 1) = \frac{1}{5}$, and we conclude that $P(Y + 2 \geq 3X) = \frac{4}{5}$.

[Method 2: If we did not notice that $X$ was uniform, we could still easily solve the problem: First notice that $P(Y + 2 \geq 3X) = P(X \geq 2) + P(X \leq 1)$, as above. Also notice $f(x) = \frac{1}{5}$ for $0 \leq x \leq 5$ and $f(x) = 0$. Then integrate, to get $P(X \geq 2) = \int_2^5 \frac{1}{5} \, dx = \frac{3}{5}$ and $P(X \leq 1) = \int_0^1 \frac{1}{5} \, dx = \frac{1}{5}$, so we conclude $P(Y + 2 \geq 3X) = \frac{4}{5}$.]
3. Southwest Airlines, having observed that 5% of the people making reservations on a flight do not show up for the flight, sells 100 tickets for a plane flight that has 95 seats. What is the approximate probability that there will be enough seats available all the people who show up for the flight? [Assume that the people act independently, i.e., assume that each person shows up—or does not show up—individually of the behavior of the other people.]

**Answer.** First of all, the exact answer is \[ \sum_{j=0}^{95} \binom{100}{j}(.95)^j(.05)^{100-j} \approx .5640. \] This can also be derived by writing \[ 1 - \sum_{j=96}^{100} \binom{100}{j}(.95)^j(.05)^{100-j} \approx .5640. \] These numbers are probably difficult and/or time consuming to get on your calculator. It is easier to do one of the following methods:

**Method 1.** (Poisson approximation.) Let \( X \) denote the number of people who do not show up for the flight. Then \( X \) is Binomial with \( p = .05 \) and \( n = 100 \), and the desired probability is \( P(X \geq 5) \). Since \( n \) is large and \( p \) is small, and \( np = (100)(.05) = 5 \) is reasonably sized, we can use a Poisson approximation, i.e., let \( Y \) be Poisson with parameter \( \lambda = np = 5 \), so the desired probability is approximately

\[
P(Y \geq 5) = 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) - P(Y = 3) - P(Y = 4) \\
= 1 - e^{-5}\left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right) = 1 - e^{-5}\frac{523}{8} \approx .5595
\]  

**Method 2.** (Normal approximation.) Let \( X \) denote the number of people who do not show up for the flight. Then \( X \) is Binomial with \( p = .05 \) and \( n = 100 \), and the desired probability is \( P(X \geq 5) \). Since \( n \) is large, we can use a Normal approximation, i.e., let \( Y \) be normal with mean \( np = 5 \) and variance \( np(1-p) = 4.75 \), so the desired probability is approximately

\[
P(X \geq 5) = P(X \geq 4.5) \approx P(Y \geq 4.5) = P\left(\frac{Y - 5}{\sqrt{4.75}} \geq \frac{4.5 - 5}{\sqrt{4.75}}\right) = P\left(Z \geq \frac{4.5 - 5}{\sqrt{4.75}}\right)
\]  
So we get

\[
P(X \geq 5) \approx P(Z \geq - .23) = P(Z \leq .23) = \Phi(.23) = .5910
\]

**Method 3.** (Normal approximation.) Let \( X \) denote the number of people who show up for the flight. Then \( X \) is Binomial with \( p = .95 \) and \( n = 100 \), and the desired probability is \( P(X \geq 5) \). Since \( n \) is large, we can use a Normal approximation, i.e., let \( Y \) be normal with mean \( np = 95 \) and variance \( np(1-p) = 4.75 \), so the desired probability is approximately

\[
P(X \leq 95) = P(X \leq 95.5) \approx P(Y \leq 95.5) = P\left(\frac{Y - 95}{\sqrt{4.75}} \leq \frac{95.5 - 95}{\sqrt{4.75}}\right) = P\left(Z \leq \frac{95.5 - 95}{\sqrt{4.75}}\right)
\]  
So we get

\[
P(X \leq 95) \approx P(Z \leq .23) = \Phi(.23) = .5910
\]
4. A machine produces bolts. The length of each bolt, in centimeters, is a normal random variable with mean 5 and standard deviation 0.2. [The lengths of the bolts are independent from each other.] A bolt is called defective if its length falls outside the interval (4.8, 5.2). What is the probability that, in a randomly selected box of 10 bolts, at most one bolt is defective? (I.e., What is the probability that zero or one bolts are defective?)

**Answer.** Let $p$ denote the probability that a randomly-selected bolt is not defective. Let $X$ denote the length of the bolt, so $p = P(4.8 \leq X \leq 5.2) = P\left(\frac{4.8-5}{0.2} \leq \frac{X-5}{0.2} \leq \frac{5.2-5}{0.2}\right) = P(-1 \leq Z \leq 1)$, where $Z$ is standard normal. So $p = P(Z \leq 1) - P(Z \leq -1) = \Phi(1) - P(Z \geq 1) = \Phi(1) - (1 - P(Z \leq 1)) = 2\Phi(1) - 1 \approx .6826$.

Then let $Y$ be a Binomial random variable with $p = .6826$ and $n = 10$. So $Y$ is the number of bolts that are not defective. So the desired probability is

$$P(Y \geq 9) = \binom{10}{9}p^9(1-p)^1 + \binom{10}{10}p^{10}(1-p)^0 \approx .1241$$
5. Let $X$ be exponentially distributed with parameter $\lambda$, i.e., with mean $1/\lambda$. For a fixed value of $t$ with $t < \lambda$, find $E[e^{tX}]$.

**Answer.** We compute

$$E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx = \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x} \, dx = \int_{0}^{\infty} \lambda e^{(t-\lambda)x} \, dx = \lambda \frac{e^{(t-\lambda)x}}{t - \lambda} \bigg|_{x=0}^{\infty} = \frac{\lambda}{\lambda - t}$$
6. On your way to work you have to drive through a traffic intersection, where you may be stopped at a traffic light.

The light turns green every five minutes, at the times 7:00 AM, 7:05 AM, 7:10 AM, etc., etc., 7:55 AM, and 8:00 AM. (The light remains green for two minutes; afterwards, it becomes red.)

The light turns red every five minutes, at the times 7:02 AM, 7:07 AM, 7:12 AM, etc., etc., 7:52 AM, 7:57 AM. (The light remains red for three minutes; afterwards, it becomes green.)

You arrive at the intersection at a time that is uniformly distributed over the interval from 7:00 AM to 8:00 AM.

If the light is green when you arrive, you do not wait.

If the light is red when you arrive, you must wait for it to turn green!

Find the expected time that you spend waiting at the light.

**Answer.** Method 1. If we arrive between 7:00 and 7:05, then our arrival time $X$ (after 7:00, measured in minutes) is uniform on the interval $[0, 5]$, our expected waiting time is $E[X] = \int_0^2 0 \frac{1}{5} dx + \int_2^5 (5 - x) \frac{1}{5} dx = -\frac{1}{10}(5 - x)^2|_{x=2} = 9/10 = 0.9$ minutes. Similarly, by a similar computation, if we arrive between 7:05 and 7:10, our expected waiting time is 0.9 minutes. If we arrive between 7:10 and 7:15, our expected waiting time is 0.9 minutes. Etc., etc. So no matter when we arrive, we expect to wait 0.9 minutes.

Method 2. We see that $2/5$ of the time, we have no wait at all. The other $3/5$ of the time, we have a uniform wait between 0 and 3 minutes, so we expect to wait 1.5 minutes in such a case. So our expected waiting time is $(2/5)(0) + (3/5)(1.5) = 0.9$ minutes.

Method 3. A direct integration is (of course) possible but a little repetitive. We let $X$ denote the waiting time, in minutes, after 7:00. Then $X$ is uniform on the interval $[0, 60]$, so the expected waiting time is

$$E[X] = \int_0^2 \frac{1}{60} dx + \int_2^5 (5 - x) \frac{1}{60} dx + \int_5^7 (0) \frac{1}{60} dx + \int_7^{10} (10 - x) \frac{1}{60} dx + \int_{10}^{12} (0) \frac{1}{60} dx + \int_{12}^{15} (15 - x) \frac{1}{60} dx + \cdots$$

The integrals from 0 to 2, and from 5 to 7, and from 10 to 12, etc., are all 0. The other integrals are each equal to $3/40$, and there are 12 such integrals, so the expected value is $(12)(3/40) = 9/10$. 


7. One thousand cars are each tested for their overall reliability. Independent of the other cars, each car passes the reliability test with probability 0.95. Use a normal approximation to compute the probability that at most 960 cars are considered reliable (i.e., 960 or less are considered reliable). Hint: Be sure to use continuity correction in your solution.

**Answer.** Let $X$ denote the number of cars that are reliable. So $X$ is Binomial with $n = 1000$ and $p = 0.95$. So $X$ is approximately normal with mean $np = 950$ and variance $np(1 - p) = \sqrt{47.5} \approx 6.89$, so the desired probability is

$$P(X \leq 960) = P(X \leq 960.5) = P\left(\frac{X - 950}{\sqrt{47.5}} \leq \frac{960.5 - 950}{\sqrt{47.5}}\right) \approx P\left(Z \leq \frac{960.5 - 950}{\sqrt{47.5}}\right).$$

So we get

$$P(X \leq 960) \approx P(Z \leq 1.52) = \Phi(1.52) = .9357$$

By the way, the exact probability is

$$P(X \leq 960) = \sum_{j=0}^{960} \binom{1000}{j} (.95)^j (.05)^{1000-j} \approx .9402$$
8. Consider $N$ players, $A_1, A_2, \ldots, A_N$ arranged in a circle, with Player $A_1$ starting the game, and play continuing around the circle.

On a player’s turn, she throws a biased coin with probability $p$ of showing heads (probability $1 - p$ of showing tails). The game continues around the circle until a head appears. The player who throws the first “head” is the winner:

- Player $A_1$ begins the game, performing the first toss.
- Player $A_2$ is next, performing the second toss.
- Player $A_3$ is next, performing the third toss.
- Etc., etc.
- Player $A_N$ is next, performing the $N$th toss.
- Player $A_1$ is next, performing the $(N + 1)$st toss.
- Player $A_2$ is next, performing the $(N + 2)$nd toss.
- Etc., etc.

Play continues until the first “head” appears; the player who throws the first “head” is the winner.

For a fixed value of $k$ with $1 \leq k \leq N$, find the probability that Player $A_k$ is the winner.

**Answer.** In order for $A_k$ to be the winner, the first $k - 1$ players must each have tails. Afterwards, there must be $jN$ consecutive rounds of tails, for some $j \geq 0$ (i.e., each person must get tails exactly $j$ times in a row). Finally, after that, player $A_k$ must get heads. So the probability that $A_k$ wins is

\[
(1 - p)^{k-1} \sum_{j=0}^{\infty} (1 - p)^{jN} p = p(1 - p)^{k-1} \sum_{j=0}^{\infty} ((1 - p)^N)^j = \frac{p(1 - p)^{k-1}}{1 - (1 - p)^N}
\]