PROBLEMS

Chapter 7 Problems

3. We compute

\[ E[|X - Y|^\alpha] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x - y|^\alpha f(x, y) \, dx \, dy \]

\[ = \int_0^1 \int_0^1 |x - y|^\alpha \, dx \, dy \]

\[ = \int_0^1 \int_y^1 (x - y)^\alpha \, dx \, dy + \int_0^1 \int_0^y (y - x)^\alpha \, dx \, dy \]

\[ = \int_0^1 \frac{(x - y)^{\alpha+1}}{\alpha + 1} \bigg|_{x=y}^1 \, dy + \int_0^1 \frac{(y - x)^{\alpha+1}}{\alpha + 1} \bigg|_0^y \, dy \]

\[ = \int_0^1 \frac{(1-y)^{\alpha+1}}{\alpha + 1} \, dy + \int_0^1 \frac{y^{\alpha+1}}{\alpha + 1} \, dy \]

\[ = \frac{(1-y)^{\alpha+2}}{(\alpha + 1)(\alpha + 2)} \bigg|_{y=0}^1 + \frac{y^{\alpha+2}}{(\alpha + 1)(\alpha + 2)} \bigg|_{y=0}^1 \]

\[ = \frac{1}{(\alpha + 1)(\alpha + 2)} + \frac{1}{(\alpha + 1)(\alpha + 2)} \]

\[ = \frac{2}{(\alpha + 1)(\alpha + 2)} \]

8. We let \( X \) denote the number of occupied tables. As in the hint, we define

\[ X_i = \begin{cases} 1 & \text{if the } i \text{th arrival sits at a previously unoccupied table} \\ 0 & \text{otherwise} \end{cases} \]

and thus \( X = X_1 + \cdots + X_N \). So \( E[X] = E[X_1 + \cdots + X_N] = E[X_1] + \cdots + E[X_N] \). Since \( X_i \) only takes on the values 0 and 1, then \( E[X_i] = P(X_i = 1) \), which is the probability that the \( i \)th arrival sits a previously unoccupied table. This happens if and only if the \( i \)th arrival is not friends with any of the first \( i - 1 \) arrivals, so

\[ E[X_i] = P(X_i = 1) = (1 - p)^{i-1} \]

Thus

\[ E[X] = \sum_{i=1}^{N} (1 - p)^{i-1} = \frac{1 - (1 - p)^N}{1 - (1 - p)} = \frac{1 - (1 - p)^N}{p} \]
9a. The number of empty urns is $X = X_1 + \cdots + X_n$ where

$$X_i = \begin{cases} 
1 & \text{if urn } i \text{ is empty} \\
0 & \text{otherwise}
\end{cases}$$

Only the balls numbered $i, i+1, i+2, \ldots, n$ can possibly go into urn $i$. The probability that ball $i$ goes into urn $i$ is $1/i$; the probability that ball $i+1$ goes into urn $i$ is $1/(i+1)$; the probability that ball $i+2$ goes into urn $i$ is $1/(i+2)$; etc., etc. So the probability that urn $i$ is empty is $(1 - \frac{1}{i}) (1 - \frac{1}{i+1}) (1 - \frac{1}{i+2}) \cdots (1 - \frac{1}{n})$, or more simply

$$\left( \frac{i-1}{i} \right) \left( \frac{i}{i+1} \right) \left( \frac{i+1}{i+2} \right) \cdots \left( \frac{n-1}{n} \right) = \frac{i-1}{n}$$

So $E[X_i] = P(X_i = 1) = \frac{i-1}{n}$. Thus $E[X] = \sum_{i=1}^{n} \frac{i-1}{n} = \frac{1}{n} \sum_{i=1}^{n} (i - 1) = \frac{1}{n} \left( \frac{(n-1)n}{2} \right) = \frac{n-1}{2}$.

9b. There is only one way that none of the urns can be empty: Namely, the $i$th ball must go into the $i$th urn for each $i$. To see this, first note that the $n$th ball is the only ball that can go into the $n$th urn. Next, there are two balls, the $n-1$ and $n$ ball that can go in urn $n-1$, but the $n$th ball is already committed to the $n$th urn, so ball $n-1$ must go into urn $n-1$. Next, there are three balls, the $n-2$, $n-1$, and $n$ ball that can go in urn $n-2$, but balls $n$ and $n-1$ are already committed to urns $n$ and $n-1$, respectively, so ball $n-2$ must go into urn $n-2$. Similar reasoning continues.

So the probability that none of the urns are empty is $(\frac{1}{n}) (\frac{1}{n-1}) (\frac{1}{n-2}) \cdots (\frac{1}{1}) = \frac{1}{n!}$.

11. The number of changeovers is $X = X_2 + \cdots + X_n$ where

$$X_i = \begin{cases} 
1 & \text{if a changeover occurs from the } (i-1)\text{st flip to the } i\text{th flip} \\
0 & \text{otherwise}
\end{cases}$$

So $E[X_i] = P(X_i = 1) = p(1-p) + (1-p)p = 2p(1-p)$. Thus $E[X] = (n-1)(2p)(1-p)$.

12a. The number of men who have a woman sitting next to them is $X = X_1 + \cdots + X_n$ where

$$X_i = \begin{cases} 
1 & \text{if the } i\text{th man has a woman sitting next to him} \\
0 & \text{otherwise}
\end{cases}$$

So $E[X_i] = P(X_i = 1)$. There are two seats on the end of the aisle where a man can sit; for each such seat, the probability that he is sitting there is $1/(2n)$, and the probability that a woman is sitting next to him afterwards is $n/(2n-1)$. So the probability a specific man sits on the left or right end, with a woman next to him, is $\frac{1}{2n} \cdot \frac{n}{2n-1} = \frac{1}{2n-1}$. There are $2n-2$ sets in the middle of the row; for each such seat, the probability that he is sitting there is $1/(2n)$, and the probability that a woman is sitting next to him afterwards (only on his left; only on his right; both sides; respectively) is $\frac{n}{2n-1} \cdot \frac{n-1}{2n-2} + \frac{n}{2n-1} \cdot \frac{n}{2n-2} + \frac{n}{2n-1} \cdot \frac{n}{2n-2} = \frac{3n}{2(2n-1)}$. So the probability that a specific man sits in the middle, with a woman next to him, is $(2n-2) \frac{1}{2n} \cdot \frac{3n}{2(2n-1)} = \frac{3(2n-2)}{4(2n-1)}$. So the total probability is $E[X_i] = P(X_i = 1) = \frac{1}{2n-1} + \frac{3(2n-2)}{4(2n-1)} = \frac{3n}{2(2n-1)}$. Thus $E[X] = n \left( \frac{3n}{2(2n-1)} \right) = \frac{n(3n-1)}{2(2n-1)}$.

12b. If the group is randomly seated at a round table, then we write $X = X_1 + \cdots + X_n$ where

$$X_i = \begin{cases} 
1 & \text{if the } i\text{th man has a woman sitting next to him} \\
0 & \text{otherwise}
\end{cases}$$
So $E[X_i] = P(X_i = 1)$. Regardless of where the $i$th man sits, the probability that a woman is sitting next to him afterwards (only on his left; only on his right; both sides; respectively) is $\frac{n}{2n-1} \frac{n-1}{2n-2} + \frac{n-1}{2n-1} \frac{n}{2n-2} + \frac{n}{2n-1} \frac{n-1}{2n-2} = \frac{3n}{2(2n-1)}$. So $E[X_i] = \frac{3n}{2(2n-1)}$. Thus $E[X] = n \left( \frac{3n}{2(2n-1)} \right) = \frac{3n}{2(2n-1)}$.

13. The number of people whose age matches their card is $X = X_1 + \cdots + X_{1000}$ where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th person’s age matches his card} \\ 0 & \text{otherwise} \end{cases}$$

So $E[X_i] = P(X_i = 1) = \frac{1}{1000}$. Thus $E[X] = 1000(1/1000) = 1$.

19a. Let $X$ denote the number of insects that are caught before the first type 1 catch. Each insect is of type 1 with probability $p_1$, and is not of type 1 with probability $1 - p_1$, independent of all other catches. So $X$ is a geometric random variable with probability of success $p_1$. So $E[X] = 1/p_1$.

19b. For $2 \leq i \leq r$, let $Y_i$ indicate whether a type $i$ insect is caught before a type 1 insect, i.e., $Y_i = 1$ if a type $i$ insect is caught before a type 1 insect, and $Y_i = 0$ otherwise. Then $E[Y_i] = P(Y_i = 1)$. Let $A_i$ denote the event that a type $i$ insect is caught for the first time on the $j$th catch, and none of the first $i - 1$ catches are type 1. So the $A_j$’s are distinct events, and $\bigcup_{j=1}^{\infty} A_j$ denotes the event that a type $i$ insect is caught before a type 1 insect.

So

$$E[Y_i] = P(Y_i = 1) = P \left( \bigcup_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} P(A_j) = \sum_{j=1}^{\infty} (1 - p_1 - p_i)^{j-1}p_i$$

and thus

$$E[Y_i] = \frac{p_i}{1 - (1 - p_1 - p_i)} = \frac{p_i}{p_1 + p_i}$$

Note that $Y_2 + \cdots + Y_r$ is the total number of types of insects that are caught before the first type 1 catch. So

$$E[Y_2 + \cdots + Y_r] = \sum_{i=2}^{r} E[Y_i] = \sum_{i=2}^{r} \frac{p_i}{p_1 + p_i}$$

26a. First, we note that $0 < \max(X_1, \ldots, X_n) < 1$. Then we note that, for $0 < a < 1$, we have $\max(X_1, \ldots, X_n) \leq a$ if and only if $X_i \leq a$ for all $i$, which—since the $X_i$’s are independent—happens with probability $\prod_{i=1}^{n} P(X_i \leq a) = \prod_{i=1}^{n} a = a^n$.

Let $Y = \max(X_1, \ldots, X_n)$. So $P(Y > y) = 1 - y^n$. We can use the fact that $E[Y] = \int_{0}^{\infty} P(Y > y) \, dy$ since $Y$ is nonnegative (see page 211), so

$$E[Y] = \int_{0}^{1} (1 - y^n) \, dy = \left[ y - \frac{y^{n+1}}{n+1} \right]_{y=0}^{1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Alternately, we can write $F_Y(y) = P(Y \leq y) = y^n$. Thus $f_Y(y) = ny^{n-1}$, and we conclude

$$E[Y] = \int_{0}^{1} y(ny^{n-1}) \, dy = \int_{0}^{1} ny^n \, dy = \frac{ny^{n+1}}{n+1} \bigg|_{y=0}^{1} = \frac{n}{n+1}$$

26b. First, we note that $0 < \min(X_1, \ldots, X_n) < 1$. Then we note that, for $0 < a < 1$, we have $\min(X_1, \ldots, X_n) > a$ if and only if $X_i > a$ for all $i$, which—since the $X_i$’s are independent—happens with probability $\prod_{i=1}^{n} P(X_i > a) = \prod_{i=1}^{n} (1 - a) = (1 - a)^n$. 


Let \( Y = \min(X_1, \ldots, X_n) \). We can use the fact that \( E[Y] = \int_0^\infty P(Y > y) \, dy \) since \( Y \) is nonnegative (see page 211), so

\[
E[Y] = \int_0^1 (1 - y)^n = \left. -\frac{(1 - y)^{n+1}}{n+1} \right|_{y=0}^1 = \frac{1}{n+1}
\]

Alternately, we can write \( F_Y(y) = P(Y \leq y) = 1 - P(Y > y) = 1 - (1 - y)^n \). Thus \( f_Y(y) = n(1 - y)^{n-1} \), and we conclude

\[
E[Y] = \int_0^1 (y)(n(1 - y)^{n-1}) \, dy = \frac{1}{n+1}
\]

using integration by parts.

34a. Let \( X_i \) indicate whether or not the \( i \)th wife sits next to her husband. In other words, define

\[
X_i = \begin{cases} 
1 & \text{if the } i \text{th wife sits next to her husband} \\
0 & \text{otherwise}
\end{cases}
\]

After the wife is seated, her husband has 19 available seats. Two of these seats are next to his wife. Since \( X_i \) only takes on the values 0 or 1, thus \( E[X_i] = P(X_i = 1) = 2/19 \). So the expected number of wives who sit next to their husbands is

\[
E \left[ \sum_{i=1}^{10} X_i \right] = \sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} 2/19 = 20/19.
\]

34b. We know that \( \text{Var} \left( \sum_{i=1}^{10} X_i \right) = \sum_{i=1}^{10} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j) \). We see that

\[
E[X_i^2] = 1^2 P(X_i = 1) + 0^2 P(X_i = 0) = 2/19 \quad \text{for each } i.
\]

Therefore \( \text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = \frac{2}{19} - \left( \frac{2}{19} \right)^2 = \frac{34}{361} \) for each \( i \). For \( i \neq j \), we notice that \( X_iX_j \) equals 1 if the \( i \)th and \( j \)th wives are each seated next to their husbands, and \( X_iX_j = 0 \) otherwise. So \( E[X_iX_j] = P(X_iX_j = 1) \). The probability that the \( i \)th wife sits next to her husband is \( 2/19 \). Given that the \( i \)th wife sits next to her husband, then 18 seats in a row remain empty. The probability that the wife sits on the end of such a row with her husband next to her is \( (2)(1/18)(1/17) \); the probability that the wife is seated strictly within the row, and her husband is next to her, is \( (16)(1/18)(2/17) \). Thus, \( E[X_iX_j] = P(X_iX_j = 1) = \frac{2}{19} \left( \frac{2}{18} + \frac{1}{17} \right) = \frac{36}{361} \). So \( \text{Cov}(X_iX_j) = E[X_iX_j] - E[X_i]E[X_j] = \frac{36}{361} - \left( \frac{2}{19} \right) \left( \frac{2}{19} \right) = \frac{36}{361} \). Thus \( \text{Var} \left( \sum_{i=1}^{10} X_i \right) = \sum_{i=1}^{10} \frac{34}{361} + 2 \sum_{i<j} \frac{36}{361} = (10) \left( \frac{34}{361} \right) + (2)(45) \left( \frac{36}{361} \right) = 360/361 \).

36. We write \( X_i \) to indicate if the \( i \)th roll is a “1” or not. Similarly, we write \( Y_i \) to indicate if the \( i \)th roll is a “2” or not. In other words,

\[
X_i = \begin{cases} 
1 & \text{if the } i \text{th roll is a “1”} \\
0 & \text{otherwise}
\end{cases} \quad Y_i = \begin{cases} 
1 & \text{if the } i \text{th roll is a “2”} \\
0 & \text{otherwise}
\end{cases}
\]

Thus \( X = \sum_{i=1}^n X_i \) and \( Y = \sum_{i=1}^n Y_i \). We also know that

\[
\text{Cov}(X,Y) = \text{Cov} \left( \sum_{i=1}^n X_i, \sum_{j=1}^n Y_j \right) = \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, Y_j)
\]

We recall \( \text{Cov}(X_i, Y_j) = E[X_iY_j] - E[X_i]E[Y_j] \).

For \( i = j \), we see that \( X_iY_j = 0 \) since the same roll cannot be both “1” and “2”. So, for \( i = j \), we have \( E[X_iY_j] = 0 \), and of course \( E[X_i] = E[Y_j] = 1/6 \), so \( \text{Cov}(X_i, Y_j) = -1/36 \).
For $i \neq j$, we see that $X_iY_j = 1$ with probability $1/36$, so $E[X_iY_j] = 1/36$, and thus $\text{Cov}(X_i, Y_j) = \frac{1}{36} - \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) = 0$. (An immediate way to see that $\text{Cov}(X_i, Y_j) = 0$ is to notice that $X_i$ and $Y_j$ are independent when $i \neq j$.)

Thus

$$\text{Cov}(X, Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, Y_j) = \sum_{i=1}^{n} \text{Cov}(X_i, Y_i) + \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(X_i, Y_j)$$

$$= \sum_{i=1}^{n} \frac{1}{36} + \sum_{i=1}^{n} \sum_{j \neq i} 0 = -n/36$$

38. Note $E[XY] = \int_{0}^{\infty} \int_{0}^{\infty} (xy) \left(\frac{2e^{-2x}}{x}\right) dy 

\text{d}x = 1/4$, and $E[X] = \int_{0}^{\infty} \int_{0}^{\infty} (x) \left(\frac{2e^{-2x}}{x}\right) dy 

\text{d}x = 1/2$ and $E[Y] = \int_{0}^{\infty} \int_{0}^{\infty} (y) \left(\frac{2e^{-2y}}{y}\right) dy 

\text{d}x = 1/4$, so $\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = 1/8$.

40. We compute $E[XY] = \int_{0}^{\infty} \int_{0}^{\infty} (xy) \left(\frac{e^{-\frac{(x+y)}{y}}}{y}\right) dx \text{d}y = 2$, and also we have $E[X] = \int_{0}^{\infty} \int_{0}^{\infty} (x) \left(\frac{e^{-\frac{(x+y)}{y}}}{y}\right) dx \text{d}y = 1$, and $E[Y] = \int_{0}^{\infty} \int_{0}^{\infty} (y) \left(\frac{e^{-\frac{(x+y)}{y}}}{y}\right) dy \text{d}x = 1$, so $\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 2 - (1)(1) = 1$.

To perform these integrations, it is helpful to note (as in class) that $\frac{1}{y}e^{-x/y}$ is the density of an exponential distribution with $\lambda = 1/y$, so $\int_{0}^{\infty} \frac{1}{y}e^{-x/y} dx = 1$ (just because $\frac{1}{y}e^{-x/y}$ is a density function) and also $\int_{0}^{\infty} (x) \frac{1}{y}e^{-x/y} dx = \frac{1}{\lambda} = y$ is the mean of the exponential distribution.

42a. Let $X_i$ indicate whether or not the $i$th pair consists of a man and a woman. In other words, define

$$X_i = \begin{cases} 
1 & \text{if the } i\text{th pair consists of a man and a woman} \\
0 & \text{otherwise}
\end{cases}$$

Since $X_i$ only takes on the values 0 or 1, thus $E[X_i] = P(X_i = 1)$. There are $\binom{20}{10}$ = 190 ways that the $i$th pair can be selected. There are $\binom{10}{1}\binom{10}{1} = 100$ ways that the pair consists of a man and a woman. So $E[X_i] = 100/190 = 10/19$. So the expected number of pairs consisting of a man and a woman is $E \left[ \sum_{i=1}^{10} X_i \right] = \sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} 10/19 = 100/19$.

We know that $\text{Var} \left( \sum_{i=1}^{10} X_i \right) = \sum_{i=1}^{10} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j)$. We see that $E[X_i^2] = 1^2 P(X_i = 1) + 0^2 P(X_i = 0) = 10/19$ for each $i$. Therefore $\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = \frac{10}{19} - \left(\frac{10}{19}\right)^2 = \frac{90}{361}$ for each $i$. For $i \neq j$, we notice that $X_iX_j$ equals 1 if the $i$th and $j$th pairs each consist of a man and a woman, and $X_iX_j = 0$ otherwise. So $E[X_iX_j] = P(X_iX_j = 1) = \binom{10}{1}\binom{8}{1} = \frac{90}{361}$. So $\text{Cov}(X_iX_j) = E[X_iX_j] - E[X_i]E[X_j] = \frac{90}{361} - \left(\frac{10}{19}\right) \left(\frac{10}{19}\right) = 10/6137$.

Thus $\text{Var} \left( \sum_{i=1}^{10} X_i \right) = \sum_{i=1}^{10} \frac{90}{361} + 2 \sum_{i<j} \frac{10}{6137} = \binom{10}{1} \left(\frac{90}{361}\right) + (2)(45) \left(\frac{10}{6137}\right) = \frac{16200}{6137}$.?
Let $Y_i$ indicate whether or not the $i$th pair consists of a married couple. In other words, define

$$Y_i = \begin{cases} 
1 & \text{if the } i\text{th pair consists of a married couple} \\
0 & \text{otherwise} 
\end{cases}$$

Since $Y_i$ only takes on the values 0 or 1, thus $E[Y_i] = P(Y_i = 1)$. There are $\binom{20}{2} = 190$ ways that the $i$th pair can be selected. There are 10 ways that the pair consists of a married couple. So $E[Y_i] = 10/190 = 1/19$. So the expected number of pairs consisting of a married couple is $E\left[\sum_{i=1}^{10} Y_i\right] = \sum_{i=1}^{10} E[Y_i] = \sum_{i=1}^{10} 1/19 = 10/19$.

We know that $\text{Var}(\sum_{i=1}^{10} Y_i) = \sum_{i=1}^{10} \text{Var}(Y_i) + 2\sum_{i<j} \text{Cov}(Y_i, Y_j)$. We see that $E[Y_i^2] = 1^2P(Y_i = 1) + 0^2P(Y_i = 0) = 1/19$ for each $i$. Therefore $\text{Var}(Y_i) = E[Y_i^2] - (E[Y_i])^2 = \frac{1}{19} - \left(\frac{1}{19}\right)^2 = \frac{18}{361}$ for each $i$. For $i \neq j$, we notice that $Y_i Y_j$ equals 1 if the $i$th and $j$th pairs each consist of a married couple, and $Y_i Y_j = 0$ otherwise. So $E[Y_i Y_j] = P(Y_i Y_j = 1)$. The probability that the $i$th pair consists of a married couple is $1/19$. Given that the $i$th pair consists of a married couple, then 9 men and 9 woman remain to form the other 9 pairs. So $\text{Var}(\sum_{i=1}^{10} Y_i) = \sum_{i=1}^{10} \frac{18}{361} + 2\sum_{i<j} \text{Cov}(Y_i, Y_j)$.

48a. The number of rolls $X$ necessary to obtain a “6” is geometric with probability of success $p = 1/6$, and thus $E[X] = 1/p = 6$.

48b. Given that $Y = 1$, we know that the first outcome is a “5”. After the first roll, the number of additional rolls necessary to obtain a “6” is geometric with probability of success $p = 1/6$, so the expected number of additional rolls is 6. So the total number of expected rolls in this conditional case is $E[X \mid Y = 1] = 1 + 6 = 7$.

48b. Given that $Y = 5$, we know that none of the first four rolls is a “5”, and the fifth role is a “5”. So $P(X = 1 \mid Y = 5) = \frac{1}{5}$, and $P(X = 2 \mid Y = 5) = \left(\frac{4}{5}\right)\left(\frac{1}{6}\right)$, and $P(X = 3 \mid Y = 5) = \left(\frac{4}{5}\right)^2\left(\frac{1}{6}\right)$, and $P(X = 4 \mid Y = 5) = \left(\frac{4}{5}\right)^3\left(\frac{1}{6}\right)$. Also $P(X = 5 \mid Y = 5) = 0$. For the remaining of the possibilities, we have $P(X = 6 \mid Y = 5) = \left(\frac{4}{5}\right)^4\left(\frac{1}{6}\right)$, and $P(X = 7 \mid Y = 5) = \left(\frac{4}{5}\right)^4\left(\frac{5}{6}\right)\left(\frac{1}{5}\right)$, and $P(X = 8 \mid Y = 5) = \left(\frac{4}{5}\right)^4\left(\frac{5}{6}\right)^2\left(\frac{1}{5}\right)$, and in general, $P(X = i \mid Y = 5) = \left(\frac{4}{5}\right)^4\left(\frac{5}{6}\right)^{i-6}\left(\frac{1}{6}\right)$ for $i \geq 6$. Thus

$$E[X \mid Y = 5] = (1)\left(\frac{1}{5}\right) + (2)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right) + (3)\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right) + (4)\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + \sum_{i=6}^{\infty} i\left(\frac{4}{5}\right)^4\left(\frac{5}{6}\right)^{i-6}\left(\frac{1}{6}\right)$$

We can pull the $\left(\frac{4}{5}\right)^4$ outside of the sum above, and then change the index of “$i$” by 5 afterwards, to obtain

$$E[X \mid Y = 5] = \frac{821}{625} + \left(\frac{4}{5}\right)^4 \sum_{i=1}^{\infty} (i+5)\left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right)$$

We note that, if $Z$ is geometric with probability of success $p = 1/6$, then the summation $\sum_{i=1}^{\infty} (i+5)\left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right)$ found in the last line above is just $E[Z+5] = E[Z] + 5 = 6 + 5$.  


Therefore, plugging this result in, we obtain

\[ E[X \mid Y = 5] = \frac{821}{625} + \left(\frac{4}{5}\right)^4 (6 + 5) = \frac{3637}{625} \]

50. We first compute the marginal density of \( Y \),

\[ f_Y(y) = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y} dx = e^{-y} \]

for \( y > 0 \) and of course \( f_Y(y) = 0 \) for \( y \leq 0 \). So the conditional density of \( X \) given \( Y \) is

\[ f_{X \mid Y}(x \mid y) = \frac{e^{-x/y}e^{-y}/y}{e^{-y}} = \frac{e^{-x/y}}{y} \]

Therefore, \( X \) conditioned on having \( Y = y \) is exponential with parameter \( \lambda = 1/y \). Now we compute the desired conditional second moment:

\[ E[X^2 \mid Y = y] = \int_{-\infty}^{\infty} x^2 f_{X \mid Y}(x \mid y) dx = \int_0^\infty (x^2) \left(\frac{e^{-x/y}}{y}\right) dx = 2y^2 \]

A different method, without computing the integral on the previous line, is to just recall that an exponential random variable with parameter \( \lambda \) has mean \( \frac{1}{\lambda} \) and variance \( \frac{1}{\lambda^2} \) and thus second moment \( \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} \). So the second moment of \( X \) conditioned on \( Y = y \) is just \( \frac{2}{\lambda^2} = 2y^2 \).

53. We write \( X \) for the number of days until the prisoner reaches freedom, and we write \( Y \) for the door he selects. Since \( E[X] = E[E[X \mid Y]] \), then

\[ E[X] = E[X \mid Y = 1]P(Y = 1) + E[X \mid Y = 2]P(Y = 2) + E[X \mid Y = 3]P(Y = 3) = (E[X] + 2)(.5) + (E[X] + 4)(.3) + (1)(.2) \]

Thus \( E[X] = 12 \).

56. Let \( X_i \) indicate whether or not the elevator stops on the \( i \)th floor. In other words, define

\[ X_i = \begin{cases} 1 & \text{if the elevator stops on the } i \text{th floor} \\ 0 & \text{otherwise} \end{cases} \]
Since $X_i$ only takes on the values 0 or 1, thus $E[X_i] = P(X_i = 1)$. We write $Y$ for the number of people who enter the elevator, so $Y$ is Poisson with parameter $\lambda = 10$. Thus

$$P(X_i = 1) = \sum_{y=0}^{\infty} P(X_i = 1 \text{ and } Y = y)$$

$$= \sum_{y=0}^{\infty} P(X_i = 1 | Y = y)P(Y = y)$$

$$= \sum_{y=0}^{\infty} \left[ 1 - \left( \frac{N - 1}{N} \right)^y \right] e^{-10} \frac{10^y}{y!}$$

$$= \sum_{y=0}^{\infty} e^{-10} \frac{10^y}{y!} - e^{-10} \sum_{y=0}^{\infty} \frac{1}{y!} 10^y \left( \frac{N - 1}{N} \right)^y$$

$$= 1 - e^{-10} e^{10(N-1)/N}$$

$$= 1 - e^{-10/N}$$

So $E[X_i] = 1 - e^{-10/N}$. So the expected number of stops is $E \left[ \sum_{i=1}^{N} X_i \right] = \sum_{i=1}^{N} E[X_i] = \sum_{i=1}^{N} (1 - e^{-10/N}) = N(1 - e^{-10/N})$.

58a. Let $X$ denote the total number of flips. We write $q = 1 - p$ as the probability of tails. Let $Y$ denote whether a head or a tail appears on the first try, by writing

$$Y = \begin{cases} H & \text{if "heads" appears on the first flip} \\ T & \text{if "tails" appears on the first flip} \end{cases}$$

Then $E[X] = E[E[X|Y]] = E[X | Y = H]P(Y = H) + E[X | Y = T]P(Y = T)$. Of course $P(Y = H) = p$ and $P(Y = T) = q$. Given that $Y = H$, then the number of additional flips until the first tail appears is geometric with probability of success $q$, so we expect $\frac{1}{q}$ additional flips. Thus $E[X | Y = H] = 1 + \frac{1}{q}$. Similarly, given that $Y = T$, then the number of additional flips until the first head appears is geometric with probability of success $p$, so we expect $\frac{1}{p}$ additional flips. Thus $E[X | Y = T] = 1 + \frac{1}{p}$. So

$$E[X] = \left( 1 + \frac{1}{q} \right) (p) + \left( 1 + \frac{1}{p} \right) (q) = 1 + \frac{p}{q} + \frac{q}{p} = \frac{1 - p + p^2}{pq}$$

Another possible method was suggested to me by Ms. Xueyao Chen this week. The expected value of $X$ can be written directly as

$$E[X] = \sum_{n=2}^{\infty} np^{n-1}q + \sum_{n=2}^{\infty} nq^{n-1}p$$

We know that the expected value of a geometric random variable with probability of success $q$ is $\frac{1}{q}$, and thus $\sum_{n=1}^{\infty} np^{n-1}q = \frac{1}{q}$. Subtracting the “$n = 1$” term on both sides yields $\sum_{n=2}^{\infty} np^{n-1}q = \frac{1}{q} - q$. Similarly, we know that the expected value of a geometric random variable with probability of success $p$ is $\frac{1}{p}$, and thus $\sum_{n=1}^{\infty} nq^{n-1}p = \frac{1}{p}$. Subtracting the
“\( n = 1 \)” term on both sides yields \( \sum_{n=2}^{\infty} nq^{n-1}p = \frac{1}{p} - p \). Therefore, putting these results together, we have

\[
E[X] = \frac{1}{q} - q + \frac{1}{p} - p = \frac{1}{q} + \frac{1}{p} - 1 = \frac{1-p + p^2}{pq}
\]

58b. The last flip lands heads if and only if the first flip lands tails. So the last flip lands heads with probability \( q \). (Remember, there are only two kinds of possibilities in this problem: either we have \( H, H, H, \ldots, H, T \) or we have \( T, T, \ldots, T, H \).)

**THEORETICAL EXERCISES**

5. As suggested, let \( X \) denote the number of the \( A_i \) that occur. Let \( X_i \) indicate if \( A_i \) occurs, i.e., \( X_i = 1 \) if \( A_i \) occurs and \( X_i = 0 \) otherwise. So \( X = \sum_{i=1}^{n} X_i \), and \( E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P(A_i) \).

Now we observe that \( C_k = 1 \) if and only if at least \( k \) of the \( A_i \)'s occur, which happens if and only if \( X \geq k \). So \( \sum_{i=1}^{n} P(C_k) = \sum_{i=1}^{n} P(X \geq k) = \sum_{0}^{n-1} P(X > k) = \sum_{0}^{\infty} P(X > k) = E[X], \) where the third equality is true since \( P(X > k) = 0 \) for \( k \geq n \).

So we proved \( \sum_{i=1}^{n} P(A_i) = E[X] = \sum_{i=1}^{n} P(C_k) \).

9. Let \( X_i \) indicate if a new run of exactly \( k \) heads begins on the \( i \)th flip, i.e., \( X_i = 1 \) if a new run of exactly \( k \) heads begins on the \( i \)th flip, or \( X_i = 0 \) otherwise. So \( E[X_i] = P(X_i = 1) \).

First we consider the trivial case where \( k = n \). In this case, there is one run of length \( k \) with probability \( p^k \), and zero runs of length \( k \) otherwise, so the expected number of runs of length \( k \) is exactly \( p^k \).

Throughout the remainder of the problem, we assume that \( k < n \). (This keeps Cases I and III distinct in the discussion below.)

Note that \( P(X_i = 1) = 0 \) for \( i > n - k + 1 \), because a run of heads that begins on flip \( n - k + 2 \) (or later) will not have time to complete \( k \) heads before the experiment ends.

Case I. For \( i = n - k + 1 \), we only require tails on the \((n-k)\)th toss, and heads on tosses \( n - k + 1, n - k + 2, \ldots, n - k + k \). So \( E[X_{n-k+1}] = P(X_{n-k+1}) = (1-p)p^k \).

Case II. For \( 1 < i < n - k + 1 \), we require tails on the \((i-1)\)st toss and the \((i+k)\) toss, and heads on tosses \( i, i+1, \ldots, i+k-1 \), so \( E[X_i] = P(X_i) = (1-p)^2p^k \).

Case III. For \( i = 1 \), we require tails on the \((k+1)\)st toss, and heads on tosses 1, 2, \ldots, \( k \), so \( E[X_1] = P(X_1) = (1-p)p^k \).

So the total expected number of runs of exactly \( k \) heads is \( E[X_1 + \cdots + X_{n-k+1}] = E[X_1] + E[X_2] + \cdots + E[X_{n-k+1}] = 2(1-p)p^k + (n-k-1)(1-p)^2p^k \).

19. We observe that \( \text{Cov}(X + Y, X - Y) = E[(X + Y)(X - Y)] - E[X + Y]E[X - Y] \), since covariance is the expected value of the product, minus the product of the expected values.

So, we compute

\[
= E[X^2 - Y^2] - (E[X] + E[Y])(E[X] - E[Y])
= E[X^2] - E[Y^2] - (E[X])^2 + (E[Y])^2
\]
Now we use the fact that $X$ and $Y$ are identically distributed, so $E[X^2] = E[Y^2]$ and $(E[X])^2 = (E[Y])^2$. Thus $\text{Cov}(X + Y, X - Y) = 0$.

22. Consider $Y = a + bX$. Then if $b > 0$, we have

$$
\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X, a + bX)}{\sqrt{\text{Var}(X)\text{Var}(a + bX)}} = \frac{\text{Cov}(X, a) + \text{Cov}(X, bX)}{\sqrt{b^2\text{Var}(X)\text{Var}(X)}} = \frac{\text{Cov}(X, a) + b\text{Cov}(X, X)}{b\text{Var}(X)} = 0 + \frac{b\text{Var}(X)}{b\text{Var}(X)} = 1.
$$

where the last equality is true since $\text{Cov}(X, a) = E[Xa] - E[X]E[a] = aE[X] - aE[X] = 0$. So we conclude that $\rho(X, Y) = 1$.

If $b < 0$, we have

$$
\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X, a + bX)}{\sqrt{\text{Var}(X)\text{Var}(a + bX)}} = \frac{\text{Cov}(X, a) + \text{Cov}(X, bX)}{\sqrt{b^2\text{Var}(X)\text{Var}(X)}} = \frac{\text{Cov}(X, a) + b\text{Cov}(X, X)}{-b\text{Var}(X)} = \frac{0 + b\text{Var}(X)}{-b\text{Var}(X)} = 1.
$$

So we conclude that $\rho(X, Y) = -1$.

33. Let $X$ denote the number of flips that are needed until a string of $r$ heads appears. Let $Y$ denote the number of flips at the beginning of the process until the first tail appears. If $Y > r$, then the string of $r$ heads appears at the start, and $X = r$ in this case. If $Y \leq r$,
then the process starts again after \( Y \) flips. The probability that \( Y = i \) is \((1 - p)p^{i-1}\). So

\[
E[X] = E[E[X|Y]]
\]

\[
= \sum_{i=1}^{\infty} E[X \mid Y = i] P(Y = i)
\]

\[
= \sum_{i=1}^{\infty} E[X \mid Y = i](1 - p)p^{i-1}
\]

\[
= \sum_{i=1}^{r} E[X \mid Y = i](1 - p)p^{i-1} + \sum_{i=r+1}^{\infty} E[X \mid Y = i](1 - p)p^{i-1}
\]

\[
= \sum_{i=1}^{r} (i + E[X])(1 - p)p^{i-1} + \sum_{i=r+1}^{\infty} r(1 - p)p^{i-1}
\]

\[
= (1 - p) \sum_{i=1}^{r} ip^{i-1} + (1 - p)E[X] \sum_{i=1}^{r} p^{i-1} + (1 - p)r \sum_{i=r+1}^{\infty} p^{i-1}
\]

\[
= (1 - p) \sum_{i=1}^{r} \frac{d}{dp} p^i + (1 - p)E[X] \frac{1 - p^r}{1 - p} + (1 - p)r \frac{p^r}{1 - p}
\]

\[
= (1 - p) \frac{d}{dp} \sum_{i=1}^{r} p^i + E[X](1 - p^r) + rp^r
\]

Which yields

\[
p^r E[X] = (1 - p) \frac{d}{dp} \frac{p - p^{r+1}}{1 - p} + rp^r
\]

or equivalently,

\[
p^r E[X] = (1 - p) \frac{1 + rp^{r+1} - rp^r - p^r}{(1 - p)^2} + rp^r
\]

i.e., \( p^r E[X] = \frac{1 - p^r}{1 - p} \), so we conclude \( E[X] = \frac{1}{p^r} \frac{1 - p^r}{1 - p} \).

34a. As suggested, let \( T_r \) denote the number of flips required to obtain a run of \( r \) consecutive heads. Then \( E[T_r \mid T_{r-1} = i] = (p)(i + 1) + (1 - p)(i + 1 + E[T_i]) \); to see this, just note that after \( r - 1 \) consecutive heads have already appeared (at the end of \( i \) tosses), one more head is sufficient to complete \( r \) consecutive tosses, which happens with probability \( p \) (resulting in a total of \( i + 1 \) tosses), or a tail appears, with probability \( 1 - p \), and the process starts again, with \( i + 1 \) tosses already having been completed. Thus, no matter what value of \( i \) is used, we have

\[
E[T_r \mid T_{r-1}] = (p)(T_{r-1} + 1) + (1 - p)(T_{r-1} + 1 + E[T_i])
\]

34b. Taking expectations on both sides, we obtain

\[
E[E[T_r \mid T_{r-1}]] = E[(p)(T_{r-1} + 1) + E[(1 - p)(T_{r-1} + 1 + E[T_i])]
\]

which simplifies to

\[
E[T_r] = (p)(E[T_{r-1}] + 1) + (1 - p)(E[T_{r-1}] + 1 + E[T_i])
\]

or equivalently \( E[T_r] = \frac{1}{p} E[T_{r-1}] + \frac{1}{p} \).
**34c.** We see that $T_1$ is a geometric random variable with probability of success $p$, so $E[T_1] = \frac{1}{p}$.

**34d.** We have $E[T_2] = \frac{1}{p^2} + \frac{1}{p}$, and $E[T_3] = \frac{1}{p^3} + \frac{1}{p^2} + \frac{1}{p}$, etc., etc., and in general $E[T_r] = \sum_{i=1}^r (1/p)^i = \frac{\frac{1}{p} - \frac{1}{p^{r+1}}}{1 - \frac{1}{p}} = \frac{1 - p^{-r}}{p-1} = \frac{p^{-r-1}}{1-p} = \frac{1 - p^r}{p^{-1} - p}$. 