PROBLEMS

We always let “Z” denote a standard normal random variable in the computations below. We simply use the chart from page 222 of the Ross book, when working with a standard normal random variable; of course, a more accurate computation is possible using the computer.

2. We first compute
\[
1 = \int_0^{\infty} Cxe^{-x/2} dx = C \left( \frac{e^{-x/2}}{-1/2} \bigg|_{x=0}^{\infty} - \int_0^{\infty} e^{-x/2} dx \right)
\]
\[
= 2C \int_0^{\infty} e^{-x/2} dx = -4Ce^{-x/2} \bigg|_{x=0}^{\infty} = 4C
\]

Thus \(C = 1/4\). Now we compute the probability that the system functions for at least 5 months:
\[
P(X \geq 5) = \int_5^{\infty} \frac{1}{4}xe^{-x/2} dx = \frac{1}{4} \left( \frac{e^{-x/2}}{-1/2} \bigg|_{x=5}^{\infty} - \int_5^{\infty} e^{-x/2} dx \right)
\]
\[
= \frac{1}{4} \left( 10e^{-5/2} + 2 \int_5^{\infty} e^{-x/2} dx \right) = \frac{1}{4} \left( 10e^{-5/2} - 4e^{-5/2} \bigg|_{x=5}^{\infty} \right) = \frac{7}{2} e^{-5/2}
\]

3a. The function
\[
f(x) = \begin{cases} 
C(2x - x^3) & \text{if } 0 < x < 5/2 \\
0 & \text{otherwise}
\end{cases}
\]

cannot be a probability density function. To see this, note \(2x - x^3 = -x(x - \sqrt{2})(x + \sqrt{2})\). If \(C = 0\), then \(f(x) = 0\) for all \(x\), so \(\int_{-\infty}^{\infty} f(x) = 0\), but every probability density function has \(\int_{-\infty}^{\infty} f(x) = 1\). If \(C > 0\), then \(f(x) < 0\) on the range \(x \in (0, \sqrt{2})\), but every probability density function has \(f(x) \geq 0\) for all \(x\). If \(C < 0\), then \(f(x) < 0\) on the range \(x \in (\sqrt{2}, 5/2)\), but every probability density function has \(f(x) \geq 0\) for all \(x\). So no value of \(C\) will satisfy all of the properties needed for a probability density function.

3b. The function
\[
f(x) = \begin{cases} 
C(2x - x^2) & \text{if } 0 < x < 5/2 \\
0 & \text{otherwise}
\end{cases}
\]
cannot be a probability density function. To see this, note \(2x - x^2 = -x(x-2)\). If \(C = 0\), then \(f(x) = 0\) for all \(x\), so \(\int_{-\infty}^{\infty} f(x) = 0\), but every probability density function has \(\int_{-\infty}^{\infty} f(x) = 1\). If \(C > 0\), then \(f(x) < 0\) on the range \(x \in (0, 2)\), but every probability density function has \(f(x) \geq 0\) for all \(x\). If \(C < 0\), then \(f(x) < 0\) on the range \(x \in (2, 5/2)\), but every probability density function has \(f(x) \geq 0\) for all \(x\). So no value of \(C\) will satisfy all of the properties needed for a probability density function.
5. Let \( X \) denote the volume of sales in a week, given in thousands of gallons. We have the density of \( X \), and we want \( a \) with \( P(X > a) = .01 \). Note that we need \( 0 < a < 1 \). We have
\[
.01 = P(X > a) = \int_{a}^{\infty} f(x) \, dx = \int_{a}^{1} 5(1-x)^4 \, dx + \int_{1}^{\infty} 0 \, dx = 5 \left( \frac{1-x}{5} \right)^{4} \bigg|_{x=a}^{1} = (1-a)^{5}
\]
So \(.01 = (1-a)^{5} \), and thus \( \sqrt[5]{.01} = 1-a \), so \( a = 1 - \sqrt[5]{.01} \approx .6019 \).

6a. We compute
\[ E[X] = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{0} (x) (0) \, dx + \int_{0}^{\infty} (x)(4) (e^{-x/2}) \, dx = \frac{1}{4} \int_{0}^{\infty} x^2 e^{-x/2} \, dx \]
\[
= \frac{1}{4} \left( x^2 e^{-x/2} \right|_{x=0}^{\infty} - \int_{0}^{\infty} 2x e^{-x/2} \, dx \right) = \frac{1}{4} \left( x^2 e^{-x/2} \right|_{x=0}^{\infty} + 4 \int_{0}^{\infty} x e^{-x/2} \, dx \right) \]
\[
= \frac{1}{4} \left( x^2 e^{-x/2} \right|_{x=0}^{\infty} + 4x e^{-x/2} \right|_{x=0}^{\infty} - 4 \int_{0}^{\infty} e^{-x/2} \, dx \right) \]
\[
= \frac{1}{4} \left( x^2 e^{-x/2} \right|_{x=0}^{\infty} + 4x e^{-x/2} \right|_{x=0}^{\infty} + 8 \int_{0}^{\infty} \, dx \right) = \frac{1}{4} (16) = 4
\]

6b. We compute
\[ 1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{-1} 0 \, dx + \int_{-1}^{1} c(1-x^2) \, dx + \int_{1}^{\infty} 0 \, dx = c \left( x - \frac{x^3}{3} \right) \bigg|_{x=-1}^{1} = c(4/3) \]
and thus \( c = 3/4 \). Now we compute
\[ E[X] = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{-1} (x) (0) \, dx + \int_{-1}^{1} \left( \frac{3}{4} - x^2 \right) \, dx + \int_{1}^{\infty} (x)(0) \, dx \]
\[ = \left( \frac{3}{4} \right) \left( x^2 - \frac{x^4}{4} \right) \bigg|_{x=-1}^{1} \]
\[ = \left( \frac{3}{4} \right) \left( \frac{1}{4} - \frac{1}{4} \right) = 0 \]

6c. We compute
\[ E[X] = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{5} (x)(0) \, dx + \int_{5}^{\infty} \frac{5}{x^2} \, dx = 5 \ln x \bigg|_{x=5}^{\infty} = +\infty \]

10a. The times in which the passenger would go to destination \( A \) are 7:05–7:15, 7:20–7:30, 7:35–7:45, 7:50–8:00, i.e., a total of 40 out of 60 minutes. So 2/3 of the time, the passenger goes to destination \( A \).

10b. The times in which the passenger would go to destination \( A \) are 7:10–7:15, 7:20–7:30, 7:35–7:45, 7:50–8:00, 8:05–8:10, i.e., a total of 40 out of 60 minutes. So 2/3 of the time, the passenger goes to destination \( A \).

11. To interpret this statement, we say that the location of the point is \( X \) inches from the left-hand-edge of the line, with \( 0 \leq X \leq L \). Since the point is chosen at random on the line, with no further clarification, it is fair to assume that the distribution is uniform, i.e., \( X \) has density function \( f(x) = \frac{1}{L} \) for \( 0 \leq x \leq L \), and \( f(x) = 0 \) otherwise.
The ratio of the shorter to the longer segment is less than $1/4$ if $X \leq \frac{1}{5}L$ or $X \geq \frac{4}{5}L$. So the ratio of the shorter to the longer segment is less than $1/4$ with probability

$$
\int_0^{\frac{1}{5}L} \frac{1}{L} \, dx + \int_{\frac{4}{5}L}^L \frac{1}{L} \, dx = 1/5 + 1/5 = 2/5
$$

14. Using proposition 2.1, we compute

$$
E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) \, dx = \int_{-\infty}^0 (x^n)(0) \, dx + \int_0^1 (x^n)(1) \, dx + \int_1^{\infty} (x^n)(0) \, dx = \int_0^1 x^n \, dx
$$

To use the definition of expectation, we need to find the density of the random variable $Y = X^n$. We first find the cumulative distribution function of $Y$. We know that $P(Y \leq a) = 0$ for $a \leq 0$, and $P(Y \leq a) = 1$ for $a \geq 1$. For $0 < a < 1$, we have $P(Y \leq a) = P(X^n \leq a) = P(X \leq \sqrt[n]{a}) = \int_0^{\sqrt[n]{a}} 1 \, dx = \sqrt[n]{a}$. Therefore, the cumulative distribution function of $Y$ is

$$
F_Y(a) = \begin{cases} 
\sqrt[n]{a} & \text{if } 0 < a < 1 \\
0 & \text{otherwise}
\end{cases}
$$

So the density of $Y$ is

$$
f_Y(x) = \begin{cases} 
\frac{1}{n} a^{\frac{1}{n} - 1} & \text{if } 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}
$$

So the expected value of $Y$ is

$$
E[Y] = \int_{-\infty}^{\infty} x f_Y(x) \, dx = \int_{-\infty}^0 (x)(0) \, dx + \int_0^1 (x) \frac{1}{n} x^{\frac{1}{n} - 1} \, dx + \int_1^{\infty} (x)(0) \, dx = \int_0^1 \frac{1}{n} x^{1/n} \, dx
$$

$$
= \left[ \frac{x^{\frac{1}{n} + 1}}{\frac{1}{n} + 1} \right]_{x=0}^{1} = \frac{1}{n + 1}
$$

15a. We compute

$$
P(X > 5) = P\left( \frac{X - 10}{6} > \frac{5 - 10}{6} \right) = P(Z > -5/6) = P(Z < 5/6) \approx \Phi(.83) \approx .7967
$$

15b. We compute

$$
P(4 < X < 16) = P\left( \frac{4 - 10}{6} < \frac{X - 10}{6} < \frac{16 - 10}{6} \right) = P(-1 < Z < 1) = P(Z < 1) - P(Z < -1) = \Phi(1) - (1 - \Phi(1)) \approx .6826
$$

15c. We compute

$$
P(X < 8) = P\left( \frac{X - 10}{6} < \frac{8 - 10}{6} \right) = P(Z < -1/3) = 1 - P(Z < 1/3) \approx 1 - \Phi(.33) \approx .3707
$$

15d. We compute

$$
P(X < 20) = P\left( \frac{X - 20}{6} < \frac{20 - 10}{6} \right) = P(Z < 5/3) \approx \Phi(1.67) \approx .9525
$$
15e. We compute

\[ P(X > 16) = P \left( \frac{X - 20}{6} > \frac{16 - 10}{6} \right) = P(Z > 1) = 1 - P(Z < 1) = 1 - \Phi(1) \approx .1587 \]

16. Let \( X \) denote the rainfall in a given year. Then \( X \) has a uniform \((\mu = 40, \sigma = 4)\) distribution, so \( P(X \leq 50) = P \left( \frac{X - 40}{4} \leq \frac{50 - 40}{4} \right) = P(Z \leq 2.5) = \Phi(2.5) \approx .9938 \), where \( Z \) has a standard normal distribution. If the rainfall in each year is independent of all other years, then it follows that the desired probability is \((P(X \leq 50))^{10} \approx (.9938)^{10} \approx .9397\).

19. We compute

\[ .10 = P(X > c) = P \left( \frac{X - 12}{2} > \frac{c - 12}{2} \right) = P \left( Z > \frac{c - 12}{2} \right) = 1 - \Phi \left( \frac{c - 12}{2} \right) \]

Thus \( \Phi \left( \frac{c - 12}{2} \right) = .90 \). So \( \frac{c - 12}{2} \approx 1.28 \). So \( c \approx 14.56 \).

20a. We have \( n = 100 \) people, each of which is in favor of a proposed rise in school taxes with probability \( p = .65 \). The number of the 100 people who are in favor of the rise in taxes is a Binomial \((n = 100, p = .65)\) random variable, which is well-approximated by a normal random variable \( X \) with mean \( np = 65 \) and variance \( np(1 - p) = 22.75 \). So the probability that at least 50 are in favor of the proposition is approximately

\[ P(X > 49.5) = P \left( \frac{X - 65}{\sqrt{22.75}} > \frac{49.5 - 65}{\sqrt{22.75}} \right) \approx P(Z > -3.25) = \Phi(3.25) \approx .9994 \]

20b. The desired probability is approximately

\[ P(59.5 < X < 70.5) = P \left( \frac{59.5 - 65}{\sqrt{22.75}} < \frac{X - 65}{\sqrt{22.75}} < \frac{70.5 - 65}{\sqrt{22.75}} \right) \approx P(-1.15 < Z < 1.15) \]

\[ = P(Z < 1.15) - P(Z < -1.15) = \Phi(1.15) - (1 - \Phi(1.15)) \approx .7498 \]

20c. The desired probability is approximately

\[ P(X < 74.5) = P \left( \frac{X - 65}{\sqrt{22.75}} < \frac{74.5 - 65}{\sqrt{22.75}} \right) \approx P(Z < 1.99) = P(Z < 1.99) \approx .9767 \]

23. Let \( X \) denote the number of times “6” shows during 1000 independent rolls of a fair die. Then \( X \) is a Binomial random variable with \( n = 1000 \) and \( p = 1/6 \). So \( X \) is approximately normal with mean \( np = 1000/6 \) and variance \( np(1 - p) = 1000(1/6)(5/6) \). Thus

\[ P(150 \leq X \leq 200) = P(149.5 \leq X \leq 200.5) \]

\[ = P \left( \frac{149.5 - (1000/6)}{\sqrt{1000(1/6)(5/6)}} \leq \frac{X - (1000/6)}{\sqrt{1000(1/6)(5/6)}} \leq \frac{200.5 - (1000/6)}{\sqrt{1000(1/6)(5/6)}} \right) \]

\[ \approx P(-1.46 \leq Z \leq 2.87) = P(Z \leq 2.87) - P(Z \leq -1.46) \]

\[ = P(Z \leq 2.87) - P(Z \geq 1.46) = P(Z \leq 2.87) - (1 - P(Z \leq 1.46)) \]

\[ = \Phi(2.87) - (1 - \Phi(1.46)) \approx .9979 - (1 - .9279) = .9258 \]

Given that “6” shows exactly 200 times, then the remaining 800 rolls are all independent, with possible outcomes 1, 2, 3, 4, 5, each appearing with probability 1/5 on each die. Let
Y denote the number of times “5” shows during the 800 rolls. Then Y is a Binomial random variable with \( n = 800 \) and \( p = 1/5 \). So Y is approximately normal with mean \( np = 800/5 = 160 \) and variance \( np(1 - p) = 800(1/5)(4/5) = 128 \). Thus

\[
P(Y < 150) = P(Y \leq 149.5) = P \left( \frac{Y - 160}{\sqrt{128}} \leq \frac{149.5 - 160}{\sqrt{128}} \right)
\]

\[
\approx P(Z \leq -0.93) = P(Z \geq 0.93) = 1 - P(Z \leq 0.93) = 1 - \Phi(0.93) \approx 1 - 0.8238 = 0.1762
\]

25. Let \( X \) denote the number of acceptable items. Then \( X \) is a Binomial random variable with \( n = 150 \) and \( p = .95 \). So \( X \) is approximately normal with mean \( np = (150)(.95) = 142.5 \) and variance \( np(1 - p) = (150)(.95)(.05) = 7.125 \). Thus

\[
P(150 - X \leq 10) = P(140 \leq X) = P(139.5 \leq X) = P \left( \frac{139.5 - 142.5}{\sqrt{7.125}} \leq \frac{X - 142.5}{\sqrt{7.125}} \right)
\]

\[
\approx P(-1.12 \leq Z) = P(Z \leq 1.12) = \Phi(1.12) = .8686
\]

26. With a fair coin, let \( X \) denote the number of heads. Then \( X \) is a Binomial random variable with \( n = 1000 \) and \( p = 1/2 \). So \( X \) is approximately normal with mean \( np = 500 \) and variance \( np(1 - p) = 250 \). Thus the probability we reach a false conclusion is

\[
P(525 \leq X) = P(524.5 \leq X) = P \left( \frac{524.5 - 500}{\sqrt{250}} \leq \frac{X - 500}{\sqrt{250}} \right) \approx P(1.55 \leq Z)
\]

\[
= 1 - P(Z \leq 1.55) = 1 - \Phi(1.55) \approx 1 - .9394 = .0506
\]

With a biased coin, let \( Y \) denote the number of heads. Then \( Y \) is a Binomial random variable with \( n = 1000 \) and \( p = .55 \). So \( Y \) is approximately normal with mean \( np = 550 \) and variance \( np(1 - p) = 247.5 \). Thus the probability we reach a false conclusion is

\[
P(Y < 525) = P(Y < 524.5) = P \left( \frac{Y - 550}{\sqrt{247.5}} \leq \frac{524.5 - 550}{\sqrt{247.5}} \right) \approx P(Z \leq -1.62)
\]

\[
= P(Z \geq 1.62) = 1 - P(Z \leq 1.62) = 1 - \Phi(1.62) \approx 1 - .9474 = .0526
\]

28. We assume that each person is left-handed, independent of all the other people. Let \( X \) denote the number of the 200 people who are lefthanded. Then \( X \) is a Binomial random variable with \( n = 200 \) and \( p = .12 \). So \( X \) is approximately normal with mean \( np = 200(.12) = 24 \) and variance \( np(1 - p) = 200(.12)(.88) = 21.12 \). Thus the probability that at least 20 of the 200 are lefthanded is

\[
P(20 \leq X) = P(19.5 \leq X) = P \left( \frac{19.5 - 24}{\sqrt{21.12}} \leq \frac{X - 24}{\sqrt{21.12}} \right) \approx P(-.98 < Z)
\]

\[
= P(Z < .98) = \Phi(.98) \approx .8365
\]

32a. Let \( X \) denote the time (in hours) needed to repair a machine. Then \( X \) is exponentially distributed with \( \lambda = 1/2 \). Then \( P(X > 2) = \int_{2}^{\infty} \frac{1}{2} e^{-(1/2)x} \, dx = e^{-1} \approx .368 \).

32b. The conditional probability is

\[
P(X > 10 \mid X > 9) = \frac{P(X > 10 \& X > 9)}{P(X > 9)} = \frac{P(X > 10)}{P(X > 9)} = \frac{\int_{10}^{\infty} \frac{1}{2} e^{-(1/2)x} \, dx}{e^{-9/2}} = e^{-5} = e^{-1/2}
\]

We could also have simply computed the line above by writing

\[
\frac{P(X > 10)}{P(X > 9)} = \frac{1 - P(X \leq 10)}{1 - P(X \leq 9)} = \frac{1 - F(10)}{1 - F(9)} = \frac{1 - (1 - e^{-(1/2)10})}{1 - (1 - e^{-(1/2)9})} = e^{-5} = e^{-1/2}
\]
An alternative method is to simply compute that probability that the waiting time would be at least one additional hour, which is 

\[ P(X > 1) = \int_1^\infty \frac{1}{2} e^{-(1/2)x} \, dx = e^{-1/2} \] (or equivalently, 

\[ P(X > 1) = 1 - P(X \leq 1) = 1 - (1 - e^{-(1/2)^1}) = e^{-1/2} \). Either way, the answer is \( e^{-1/2} \approx .6065 \).

33. Let \( X \) denote the time in years that the radio continues to function for Jones. The desired probability is 

\[ P(X > 8) = \int_8^\infty \frac{1}{8} e^{-(1/8)x} \, dx = \left. \frac{1}{8} e^{-(1/8)x} \right|_8^\infty = e^{-1} \approx .368 \]

We could also have written \( P(X > 8) = 1 - P(X \leq 8) = 1 - (1 - e^{-(1/8)^8}) = e^{-1} \approx .368 \).

37a. We compute \( P(|X| > 1/2) = P(X > 1/2 \text{ or } X < -1/2) = P(X > 1/2) + P(X < -1/2) = \frac{1}{2} + \frac{1}{2} = 1/2 \).

37b. We first compute the cumulative distribution function of \( Y = |X| \). For \( a \leq 0 \), we have \( P(Y \leq a) = 0 \), since \(|X|\) is never less than \( a \) in this case. For \( a \geq 1 \), we have \( P(Y \leq a) = 1 \), since \(|X|\) is always less than \( a \) in this case. For \( 0 < a < 1 \), we compute 

\[ P(Y \leq a) = P(|X| \leq a) = P(-a \leq X \leq a) = \frac{2a}{2} = a \]

or equivalently, \( P(Y \leq x) = x \) for \( 0 < x < 1 \). In summary, the cumulative distribution function of \( Y = |X| \) is

\[ F_Y(x) = \begin{cases} 
0 & x \leq 0 \\
1 & 0 < x < 1 \\
1 & x \geq 1 
\end{cases} \]

Differentiating throughout with respect to \( x \) yields the density of \( Y = |X| \) is

\[ f_Y(x) = \begin{cases} 
1 & 0 < x < 1 \\
0 & \text{otherwise} 
\end{cases} \]

Intuitively, if \( X \) is uniformly distributed on \((-1, 1)\), then \( Y = |X| \) is uniformly distribution on \([0, 1)\).

39. Since \( X \) is exponentially distributed with \( \lambda = 1 \), then \( X \) has density \( f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\
0 & \text{otherwise} \end{cases} \) and cumulative distribution function 

\[ F_X(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\
0 & \text{otherwise} \end{cases} \]

Next, we compute the cumulative distribution function of \( Y = \log X \). We have 

\[ P(Y \leq a) = P(\log X \leq a) = P(X \leq e^a) = F_X(e^a) = 1 - e^{-e^a} \]

or equivalently, \( P(Y \leq x) = 1 - e^{-e^x} \). Differentiating throughout with respect to \( x \) yields the density of \( Y = \log X \) is

\[ f_Y(x) = e^x e^{-e^x} \]
40. Since $X$ is uniformly distributed on $(0, 1)$, then $X$ has density

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and cumulative distribution function

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Next, we compute the cumulative distribution function of $Y = e^X$. For $a \leq 1$, we have $P(Y \leq a) = 0$, since $Y = e^X$ is never less than $a$ in this case (i.e., $X$ is never less than 0 in this case). For $a \geq e$, we have $P(Y \leq a) = 1$, since $Y = e^X$ is always less than $a$ in this case (i.e., $X$ is always less than 0 in this case). For $1 < a < e$ (and thus $0 < \ln a < 1$), we compute

$$P(Y \leq a) = P(e^X \leq a) = P(X \leq \ln a) = F_X(\ln a) = \ln a$$

or equivalently, $P(Y \leq x) = \ln x$ for $1 < x < e$. In summary, the cumulative distribution function of $Y = e^X$ is

$$F_Y(x) = \begin{cases} 0 & x \leq 1 \\ \ln x & 1 < x < e \\ 1 & x \geq e \end{cases}$$

Differentiating throughout with respect to $x$ yields the density of $Y = e^X$ is

$$f_Y(x) = \begin{cases} \frac{1}{x} & 1 < x < e \\ 0 & \text{otherwise} \end{cases}$$

THEORETICAL EXERCISES

2. We observe that

$$E[Y] = \int_{-\infty}^{\infty} x f_Y(x) \, dx = \int_{-\infty}^{0} x f_Y(x) \, dx + \int_{0}^{\infty} x f_Y(x) \, dx$$

The second integral can be simplified by observing that $x = \int_{0}^{x} 1 \, dy$, so

$$\int_{0}^{\infty} x f_Y(x) \, dx = \int_{0}^{\infty} \int_{0}^{x} f_Y(x) \, dy \, dx = \int_{0}^{\infty} \int_{y}^{\infty} f_Y(x) \, dx \, dy = \int_{0}^{\infty} P(Y > y) \, dy$$

where the second equality in the line above follows by interchanging the order of integration over all $x$ and $y$ with $0 \leq y \leq x$.

Similarly, $x = \int_{-\infty}^{0} 1 \, dy = \int_{-\infty}^{0} -1 \, dy$, so

$$\int_{-\infty}^{0} x f_Y(x) \, dx = -\int_{-\infty}^{0} \int_{0}^{-x} f_Y(x) \, dy \, dx = -\int_{0}^{\infty} \int_{-\infty}^{-y} f_Y(x) \, dx \, dy = -\int_{0}^{\infty} P(Y < -y) \, dy$$

where the second equality in the line above follows by interchanging the order of integration over all $x$ and $y$ with $0 \leq y \leq -x$ or equivalently $x \leq -y \leq 0$. 
Altogether, this yields
\[ E[Y] = \int_0^\infty P(Y > y) \, dy - \int_0^\infty P(Y < -y) \, dy \]
as desired.

6. Let \( X \) be uniform on the interval \((0, 1)\). For each \( a \) in the range \( 0 < a < 1 \), define \( E_a = P(X \neq a) \). So \( P(E_a) = 1 \) for each \( a \). Also, \( \bigcap_{0 < a < 1} E_a = \emptyset \), so \( P\left( \bigcap_{0 < a < 1} E_a \right) = P(\emptyset) = 0 \).

8. We see that
\[ E[X^2] = \int_0^c x^2 f(x) \, dx \leq \int_0^c c x f(x) \, dx = c E[X] \]
Thus
\[ \text{Var}(X) = E[X^2] - (E[X])^2 \leq cE[X] - (E[X])^2 = E[X](c - E[X]) = c^2 \frac{E[X]}{c} \left( 1 - \frac{E[X]}{c} \right) \]
or equivalently, writing \( \alpha = \frac{E[X]}{c} \), we have
\[ \text{Var}(X) = c^2 \alpha (1 - \alpha) \]
We notice that \( 0 \leq E[X] = \int_0^c x f(x) \, dx \leq \int_0^c c f(x) \, dx = c \), so \( 0 \leq E[X] \leq c \), so \( 0 \leq \frac{E[X]}{c} \leq 1 \), or equivalently, \( 0 \leq \alpha \leq 1 \).

The largest value that \( \alpha (1 - \alpha) \) can achieve for \( 0 \leq \alpha \leq 1 \) is \( 1/4 \), which happens when \( \alpha = 1/2 \) (to see this, just differentiate \( \alpha (1 - \alpha) \) with respect to \( \alpha \), set the result equal to 0, and solve for \( \alpha \), which yields \( \alpha = 1/2 \); don’t forget to also check the endpoints, namely \( \alpha = 0 \) and \( \alpha = 1 \)).

Since \( \text{Var}(X) \leq c^2 \alpha (1 - \alpha) \) and \( \alpha (1 - \alpha) \leq 1/4 \), it follows that \( \text{Var}(X) \leq c^2/4 \), as desired.

12a. If \( X \) is uniform on the interval \((a, b)\), then the median is \((a + b)/2\). To see this, write \( m \) for the median, and solve \( 1/2 = F(m) = \int_a^m \frac{1}{b-a} \, dx = \frac{m-a}{b-a} \) which immediately yields \( m = \frac{(a+b)}{2} \).

12b. If \( X \) is normal with mean \( \mu \) and variance \( \sigma^2 \), then the median is \( \mu \). To see this, write \( m \) for the median, and solve \( 1/2 = F(m) = P(X \leq m) = P\left( \frac{X-\mu}{\sigma} \leq \frac{m-\mu}{\sigma} \right) \) = \( \Phi\left( \frac{m-\mu}{\sigma} \right) \) where \( Z \) is standard normal, and thus \( \frac{m-\mu}{\sigma} = 0 \), so \( m = \mu \).

12c. If \( X \) is exponential with parameter \( \lambda \), then the median is \( \frac{\ln 2}{\lambda} \). To see this, write \( m \) for the median, and solve \( 1/2 = F(m) = 1 - e^{-\lambda m} \), so \( e^{-\lambda m} = 1/2 \), so \( -\lambda m = \ln(1/2) \), so \( \lambda m = \ln 2 \), and thus \( m = \frac{\ln 2}{\lambda} \).

13a. If \( X \) is uniform on the interval \((a, b)\), then any value of \( m \) between \( a \) and \( b \) is equally valid to be used as the mode, since the density is constant on the interval \((a, b)\).

13b. If \( X \) is normal with mean \( \mu \) and variance \( \sigma^2 \), then the mode is \( \mu \). To see this, write \( m \) for the mode, and solve \( 0 = \frac{d}{dx} \frac{1}{\sqrt{2\pi} \sigma} e^{-(x-\mu)^2/(2\sigma^2)} \), i.e., \( 0 = \left( \frac{1}{\sqrt{2\pi} \sigma} e^{-(x-\mu)^2/(2\sigma^2)} \right) \frac{d}{dx} \left( \frac{1}{2\sigma} (x-\mu) \right) \), so \( x = \mu \) is the location of the mode.
If $X$ is exponential with parameter $\lambda$, then the mode is 0. To see this, note that

$$\frac{d}{dx} \lambda e^{-\lambda x} = \lambda e^{-\lambda x} - \lambda x e^{-\lambda x}$$

is the slope of the density for all $x > 0$, so the density is always decreasing for $x > 0$. So the mode must occur on the boundary of the positive portion of the density, i.e., at $x = 0$.

Consider an exponential random variable $X$ with parameter $\lambda$. To show that $Y = cX$ is exponential with parameter $\lambda/c$, it suffices to show that $Y$ has cumulative density function

$$F_Y(a) = \begin{cases} 1 - e^{-(\lambda/c)a} & a > 0 \\ 0 & \text{otherwise} \end{cases}$$

To see this, first we note that $X$ is always nonnegative, so $Y = cX$ is always nonnegative, so $F_Y(a) = 0$ for $a \leq 0$.

For $a > 0$, we check

$$F_Y(a) = P(Y \leq a) = P(cX \leq a) = P(X \leq a/c) = 1 - e^{-\lambda(a/c)} = 1 - e^{-(\lambda/c)a}$$

as desired.

Thus, $Y = cX$ has the cumulative distribution function of an exponential random variable with parameter $\lambda/c$, so $Y = cX$ must indeed be exponential with parameter $\lambda/c$.

We prove that, for an exponential random variable $X$ with parameter $\lambda$,

$$E[X^k] = \frac{k!}{\lambda^k}$$

for $k \geq 1$.

To see this, we use proof by induction on $k$.

For $k = 1$, we use integration by parts with $u = x$ and $dv = \lambda e^{-\lambda x} dx$, and thus $du = dx$ and $v = \frac{\lambda e^{-\lambda x}}{-\lambda}$, to see that

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} \, dx = \left( x \left( \frac{\lambda e^{-\lambda x}}{-\lambda} \right) \right) \bigg|_0^\infty - \int_0^\infty \frac{\lambda e^{-\lambda x}}{-\lambda} \, dx$$

or, more simply,

$$E[X] = -\frac{x}{e^{\lambda x}} \bigg|_0^\infty + \int_0^\infty e^{-\lambda x} \, dx$$

We see that the integral evaluates to $\left. \frac{e^{-\lambda x}}{-\lambda} \right|_0^\infty = \frac{1}{\lambda}$. We also see that, in the first part of the expression, plugging in $x = 0$ yields 0; using L'Hospital’s rule as $x \to \infty$ yields

$$\lim_{x \to \infty} \frac{x}{e^{\lambda x}} = \lim_{x \to \infty} \frac{1}{\lambda e^{\lambda x}} = 0$$

So we conclude that

$$E[X] = \frac{1}{\lambda}$$

This completes the base case, i.e., the case $k = 1$.

Now we do the inductive step of the proof. For $k \geq 2$, we assume that $E[X^{k-1}] = \frac{(k-1)!}{\lambda^{k-1}}$ has already been proved, and we prove that $E[X^k] = \frac{k!}{\lambda^k}$.
To do this, we use integration by parts with \( u = x^k \) and \( dv = \lambda e^{-\lambda x} \, dx \), and thus \( du = kx^{k-1} \, dx \) and \( v = \frac{\lambda e^{-\lambda x}}{-\lambda} \), to see that

\[
E[X^k] = \int_0^\infty x^k \lambda e^{-\lambda x} \, dx = (x^k) \left( \frac{\lambda e^{-\lambda x}}{-\lambda} \right) \bigg|_{x=0}^\infty - \int_0^\infty (kx^{k-1}) \left( \frac{\lambda e^{-\lambda x}}{-\lambda} \right) \, dx
\]

or, more simply,

\[
E[X^k] = -\frac{x^k}{e^{\lambda x}} \bigg|_{x=0}^\infty + \frac{k}{\lambda} \int_0^\infty x^{k-1} \lambda e^{-\lambda x} \, dx
\]

We see that the integral is \( \frac{k}{\lambda} E[X^{k-1}] \), which is (by the inductive assumption) equal to \( \frac{k}{\lambda} \frac{(k-1)!}{\lambda^{k-1}} = \frac{k!}{\lambda^k} \). We also see that, in the first part of the expression, plugging in \( x = 0 \) yields 0; using L'Hospital's rule \( k \times \) times as \( x \to \infty \) yields

\[
\lim_{x \to \infty} \frac{x^k}{e^{\lambda x}} = \lim_{x \to \infty} \frac{kx^{k-1}}{\lambda e^{\lambda x}} = \lim_{x \to \infty} \frac{k(k - 1)x^{k-2}}{\lambda^2 e^{\lambda x}} = \cdots = \lim_{x \to \infty} \frac{k!}{\lambda^k e^{\lambda x}} = 0
\]

So we conclude that

\[
E[X^k] = \frac{k!}{\lambda^k}
\]

This completes the inductive case, which completes the proof by induction.