The problem discussed at the end of the day on Wednesday was the following: Say 50 people enter a room; all of them check their hats upon arriving. Upon leaving the party in a drunken stupor, each person randomly grabs a hat. What is the probability that none of the 50 people gets her or his own hat back?

Here is the general method of solution. I will use “N” instead of 50, so that you all can see the most general solution.

We write $E_i$ to denote the event that the $i$th person gets her or his own hat back. So

$$P(\text{nobody gets her own hat back}) = 1 - P(\text{at least one person gets her own hat back})$$

$$= 1 - P\left(\bigcup_{i=1}^{N} E_i\right)$$

We focus our efforts on calculating $P\left(\bigcup_{i=1}^{N} E_i\right)$ by using inclusion-exclusion. We have

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) - \cdots + (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cap \cdots \cap E_{i_r}) + \cdots + (-1)^{N+1} P(E_1 \cap \cdots \cap E_N)$$

There are $\binom{N}{1} = N$ terms of the form $P(E_i)$, and each is equal to $\frac{1}{N}$. There are $\binom{N}{2} = \frac{N(N-1)}{2}$ terms of the form $P(E_{i_1} \cap E_{i_2})$, and each is equal to $\frac{1}{N} \cdot \frac{1}{N-1}$. There are $\binom{N}{3} = \frac{N(N-1)(N-2)}{3!}$ terms of the form $P(E_{i_1} \cap E_{i_2} \cap E_{i_3})$, and each is equal to $\frac{1}{N} \cdot \frac{1}{N-1} \cdot \frac{1}{N-2}$. Etc., etc. There are $\binom{N}{r} = \frac{N(N-1)(N-2) \cdots (N-r+1)}{r!}$ terms of the $r$th type, and each is equal to $\frac{1}{N} \cdot \frac{1}{N-1} \cdot \frac{1}{N-2} \cdots \frac{1}{N-r+1}$. For the final term we have $P(E_1 \cap \cdots \cap E_N) = \frac{1}{N} \cdot \frac{1}{N-1} \cdots \frac{1}{N-r+1}$. So we get

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \frac{1}{N} - \frac{N(N-1)}{2} \cdot \frac{1}{N} \cdot \frac{1}{N-1} + \frac{N(N-1)(N-2)}{3!} \cdot \frac{1}{N} \cdot \frac{1}{N-1} \cdot \frac{1}{N-2} - \cdots$$

$$= 1 - \frac{1}{2} + \frac{1}{3!} \cdots = \sum_{i=1}^{N} \frac{(-1)^{i+1}}{i!}$$

So

$$P(\text{nobody gets her own hat back}) = 1 - \sum_{i=1}^{N} \frac{(-1)^{i+1}}{i!} = 1 + \sum_{i=1}^{N} \frac{(-1)^i}{i!} = \sum_{i=0}^{N} \frac{(-1)^i}{i!}$$

Although the $(N+1)$st and $(N+2)$nd and $(N+3)$rd, etc., etc., terms are missing, these are all small, so we would get a very good approximation by writing

$$P(\text{nobody gets her own hat back}) \approx \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = e^{-1} \approx 0.36788$$