Consider $X, Y$ with joint distribution $f_{X,Y}(x, y) = k(5-x)(7-y)$ on the triangle with vertices at $(0,0)$, $(5,0)$, $(0,7)$, as shown in figure 1.

We need $1 = \int_0^5 \int_0^7 k(5-x)(7-y) \, dy \, dx$, so we get $k = 24/6125$.

To find the conditional probability $P(X \leq 3 \mid Y = 1)$, we need the density $f_Y(y)$ evaluated at $y = 1$.

The density of $Y$ is

$$f_Y(y) = \int_0^{\frac{5}{7}(7-y)} \frac{24}{6125} (5-x)(7-y) \, dx = \frac{24}{6125} \left( \frac{175}{2} - \frac{25y}{2} - \frac{25y^2}{14} + \frac{25y^3}{98} \right).$$

So $f_Y(1) = \frac{24}{6125} \frac{3600}{49}$. So the condition density of $X$, given $Y = 1$, is

$$f_{X \mid Y}(x \mid 1) = \frac{f_{X,Y}(x, 1)}{f_Y(1)} = \frac{24/6125 (5-x)(7-1)}{24/6125 \frac{3600}{49}} = \frac{49}{600}(5-x) \quad \text{for} \ 0 \leq x \leq \frac{5}{7}(7-1) = \frac{30}{7},$$

and $f_{X \mid Y}(x \mid 1) = 0$ otherwise.

So the desired probability is

$$P(X \leq 3 \mid Y = 1) = \int_0^3 f_{X \mid Y}(x \mid 1) = \int_0^3 \frac{49}{600}(5-x) \, dx = \frac{343}{400} = 0.8575.$$

Figure 1: The region where $X, Y$ are in the triangle.
Now we switch variables.

To find the conditional probability \( P(Y \leq 3 \mid X = 1) \), we need the density \( f_X(x) \) evaluated at \( x = 1 \).

The density of \( X \) is

\[
f_X(x) = \int_{-7}^{7-x} \frac{24}{6125} (5-x)(7-y) \, dy = \frac{24}{6125} \left( \frac{245}{2} - \frac{49x}{2} - \frac{49x^2}{10} + \frac{49x^3}{50} \right).
\]

So \( f_X(1) = \frac{24}{6125} \cdot \frac{2352}{25} \). So the condition density of \( Y \), given \( X = 1 \), is

\[
f_{Y|X}(y \mid 1) = \frac{f_{X,Y}(1, y)}{f_X(1)} = \frac{24}{6125} \cdot \frac{24}{24} \cdot \frac{2352}{25} = \frac{25}{588}(7-y) \quad \text{for } 0 \leq y \leq 7 - \frac{7}{5}(1) = \frac{28}{5},
\]

and \( f_{Y|X}(y \mid 1) = 0 \) otherwise.

So the desired probability is

\[
P(Y \leq 3 \mid X = 1) = \int_0^3 f_{Y|X}(y \mid 1) \, dy = \int_0^3 \frac{25}{588}(7-y) \, dy = \frac{275}{392} = 0.7015.
\]

***********************************************

Now we switch the entire problem around altogether. Suppose now that the joint density is \textbf{CONSTANT} on the triangle. So \( f_{X,Y}(x, y) = \frac{2}{35} \) for \( X, Y \) in the triangle, and \( f_{X,Y}(x, y) = 0 \) otherwise. Then we have no need to integrate. We can just write:

Given \( Y = 1 \), then \( 0 \leq X \leq \frac{5}{7}(7-1) = \frac{30}{7} \). Since the joint density \( f_{X,Y}(x, y) \) is constant, then the conditional density of \( X \) (given \( Y = 1 \)) is constant on this line. Thus

\[
P(X \leq 3 \mid Y = 1) = \frac{\text{length of } [0,3]}{\text{length of } [0,30/7]} = \frac{7}{10} = 0.7.
\]

Given \( X = 1 \), then \( 0 \leq Y \leq 7 - \frac{7}{5}(1) = \frac{28}{5} \). Since the joint density \( f_{X,Y}(x, y) \) is constant, then the conditional density of \( Y \) (given \( X = 1 \)) is constant on this line. Thus

\[
P(Y \leq 3 \mid X = 1) = \frac{\text{length of } [0,3]}{\text{length of } [0,28/5]} = \frac{15}{28} = 0.5357.
\]

Also, as asked near the end of class, what is \( P(Y \leq 3 \mid X \leq 1) \), when the joint density is constant? We can just calculate the ratio of the areas:

\[
P(Y \leq 3 \mid X \leq 1) = \frac{P(Y \leq 3 \text{ and } X \leq 1)}{P(X \leq 1)}.
\]

The area where \( Y \leq 3 \) and \( X \leq 1 \) is just \((3)(1) = 3\). As in basic calculus, the area where \( X \leq 1 \) is \( \int_0^1 (7 - \frac{7}{5}x) \, dx = \frac{63}{10} \). So

\[
P(Y \leq 3 \mid X \leq 1) = \frac{3}{\frac{63}{10}} = \frac{10}{21} = 0.4762.
\]