Chapter 32 Answers

1. Bob makes pots out of clay. The clay costs $5.00 per pound. Each pot is made from a random amount $X$ of clay (in pounds), uniformly distributed between 1.9 and 2.7 pounds. There is a charge to bake the pot in a kiln too (a kiln is a big oven for making clay get very hard). The charge to use the kiln to harden a pot is $0.40.

What is the probability that Bob spends more than $12.00 altogether (cost of clay plus cost of using the kiln) to make a pot of clay?

The charge to make a pot of clay is $5.00X + 0.40$ (in dollars). Since $X$ is uniform on $[1.9, 2.7]$, it follows that $Y = 5.00X + 0.40$ is uniform on $[5(1.9) + 0.40, 5(2.7) + 0.40] = [9.9, 13.9]$. So the probability that $Y$ is more than 12 is $\frac{13.9 - 12}{13.9 - 9.9} = \frac{1}{4} = 0.25$.

2. In a crayon factory, wax is rolled into cylinders, each of which are exactly 3.6 inches long, but the radius (in inches) is a uniform random variable $X$ on the interval $[.15, .17]$.

Find the probability that the volume of a crayon exceeds .30 cubic inches. Hint: The volume of a cylinder is $\pi r^2 \ell$ where $r$ is the radius and $\ell$ is the length.

If $V$ denotes the volume, in cubic inches, then

$$P(V > .30) = P((\pi)(X^2)(3.6) > .30) = P \left( X > \frac{\sqrt{.30}}{(3.6\pi)} \right) = P(X > .1629),$$

but $X$ is uniform on $[.15, .17]$, so $P(X > .1629) = \frac{.17 - .1629}{.17 - .15} = .355$.

3. Generalize Example 32.4 as follows:

Let $X$ be a uniform random variable on the interval $(0, 1)$, i.e., $X$ is uniformly distributed with $0 < X < 1$. Let $Y = X^n$ where $n$ is any positive integer.

Now find $E[Y]$.

First of all $0 < Y < 1$ always. So $f_Y(y) = 0$ if $y$ is not in the interval $(0, 1)$.

Now consider $y$ in the interval $(0, 1)$. The cumulative distribution function of $Y$ is

$$F_Y(y) = P(Y \leq y) = P(X^n \leq y) = P(X \leq y^{1/n}) = \frac{y^{1/n} - 0}{1 - 0} = y^{1/n},$$

so the density of $Y$ is $f_Y(y) = (1/n)y^{(1-n)/n}$. So the expected value of $Y$ is

$$E[Y] = \int_0^1 (y)(1/n)y^{(1-n)/n} \, dy = (1/n) \int_0^1 y^{1/n} \, dy = (1/n) \frac{y^{(1+n)/n}}{(1+n)/n} \bigg|_{y=0}^1 = \frac{1}{1 + n}.$$

4. Consider an exponential random variable $X$ with parameter $\lambda > 0$. Is it always true that, if $a$ and $b$ are positive constants, then $Y = aX + b$ is an exponential random variable too?
If your answer is “yes”, then give a justification (e.g., give an argument in favor).
If your answer is “no”, then give a very concrete counterexample (e.g., for at least one specific $a$ and $b$ of your choice, show that $Y = aX + b$ is not exponential).

If $b \neq 0$ then $Y = aX + b$ is not an exponential random variable. For instance, if $b = 3$, then $Y > 3$ always, so the density of $Y$ is nonzero on the interval $(3, \infty)$, but exponential random variables have nonzero density on $(0, \infty)$, so $Y$ is not exponential. In general, if $b \neq 0$, then $Y > b$, so the density of $Y$ is nonzero on the interval $(b, \infty)$, but exponential random variables have nonzero density on $(0, \infty)$. Thus, if $b \neq 0$, then $Y$ is not exponential!

If $b = 0$ then $Y$ is exponential. E.g., if $a = 3$, then $Y = 3X$, so the CDF of $Y$ (for $y > 0$) is

$$P(Y \leq y) = P(3X \leq y) = P(X \leq y/3) = 1 - e^{-\lambda(y/3)} = 1 - e^{-(\lambda/3)(y)},$$

so $Y$ is exponential with parameter $\lambda/3$. In general, if $b = 0$ and $a > 0$, then $Y = aX$ is exponential with parameter $\lambda/a$, because

$$P(Y \leq y) = P(aX \leq y) = P(X \leq y/a) = 1 - e^{-\lambda(y/a)} = 1 - e^{-(\lambda/a)(y)}.$$

**Review question: 5.** A man and a woman arrive at a certain location; whoever arrives first will wait for the other to arrive. Their arrival times are independent and are each uniformly distributed between noon and 1 PM. Find the probability that the first person to arrive must wait 10 minutes or longer for the second person to arrive.

Let $X$ and $Y$ (respectively) denote that number of minutes after noon until the man and woman (respectively) arrive, so $X$ and $Y$ are each uniform on $[0, 60]$. The point $(X, Y)$ is uniformly distributed in the square in Figure 1, which has total area $(60)(60) = 3600$. In the upper triangle, $Y \geq X + 10$, so the woman arrives 10 minutes or more after the man. In the lower triangle, $Y \leq X - 10$, so $X \geq Y + 10$, so the man arrives 10 minutes or more after the woman. Each triangle has area $(50)(50)/2 = 1250$. So the total desired area is $1250 + 1250 = 2500$. So the desired probability is $2500/3600 = 25/36$.

![Figure 1: Arrival time $X$ (for a man) and $Y$ (for a woman); each are uniform on $[0, 60]$](image)