Chapter 31 Answers

1. Bob is waiting for his girlfriend Alice to call. His expected wait time $X$, given in hours, has density $f_X(x) = 5e^{-5x}$ for $x > 0$, and $f_X(x) = 0$ otherwise. His expected waiting time is $E[X] = 1/5$, i.e., 1/5 of an hour, i.e., 12 minutes.

What is the probability that he must wait more than 12 minutes for her to call?

The probability he waits more than 1/5 of an hour is

$$P(X > 1/5) = \int_{1/5}^{\infty} 5e^{-5x} \, dx = -e^{-5x}\bigg|_{x=1/5}^{\infty} = e^{-1} \approx 0.367879$$

We can generalize the result above. Suppose that $Y$ is any exponential random variable with parameter $\lambda > 0$. The expected value of $Y$ is $E[Y] = 1/\lambda$. What is the probability that $Y$ is bigger than its expected value? I.e., what is $P(Y > 1/\lambda)$?

The probability of $Y > 1/\lambda$ is

$$P(Y > 1/\lambda) = \int_{1/\lambda}^{\infty} \lambda e^{-\lambda x} \, dx = -e^{-\lambda x}\bigg|_{x=1/\lambda}^{\infty} = e^{-1} \approx 0.367879$$

2. Bob is waiting for his girlfriend Alice to call. His expected wait time $X$, given in hours, has density $f_X(x) = 5e^{-5x}$ for $x > 0$, and $f_X(x) = 0$ otherwise. Find the unique time “$a$” in hours such that $P(X \leq a) = 1/2$ and $P(X > a) = 1/2$.

The unique time “$a$” satisfies 1/2 = $P(X > a) = e^{-5a}$, so $\ln(1/2) = -5a$, so $a = (1/5)\ln(2) \approx 0.138629$. Another way to solve for “$a$” is 1/2 = $P(X \leq a) = 1 - e^{-5a}$, so $e^{-5a} = 1/2$, and then the computation proceeds exactly as before, so $a = (1/5)\ln(2)$.

We can again generalize the result above. Suppose that $Y$ is any exponential random variable with parameter $\lambda > 0$. What is the unique value “$a$” such that $P(Y \leq a) = 1/2$ and $P(Y > a) = 1/2$? [This value “$a$” is called the “median”.

Again, “$a$” satisfies 1/2 = $P(X > a) = e^{-\lambda a}$, so $\ln(1/2) = -\lambda a$, so $a = (1/\lambda)\ln(2)$. Also, “$a$” satisfies 1/2 = $P(X \leq a) = 1 - e^{-\lambda a}$, so $e^{-\lambda a} = 1/2$, and thus $a = (1/\lambda)\ln(2)$.

3. Dan, Dominic, and Doug are waiting together in the living room for their girlfriends, Sally, Shellie, and Susanne, to call. Their waiting times (in hours) are independent exponential random variables, with parameters 2.1, 3.7, and 5.5, respectively. What is the probability that the phone will ring (i.e., the first call will arrive) within the next 1/10 of an hour?

If we let $X_1, X_2, X_3$ denote the waiting time until Sally, Shellie, and Susanne call, respectively, then $Y = \min(X_1, X_2, X_3)$ is the time until the phone rings for the very first time. So
Y is exponential with parameter $\lambda = 2.1 + 3.7 + 5.5 = 11.3$. Thus

$$P(Y \leq 1/10) = \int_0^{1/10} 11.3e^{-11.3y}dy = -e^{-11.3y}\bigg|_{y=0}^{1/10} = 1 - e^{-(11.3)(1/10)} \approx 0.6769667$$

4. Consider an exponential random variable $X$ with parameter $\lambda > 0$. Let $Y = \lfloor X \rfloor$, which means we get $Y$ by rounding $X$ down to the nearest integer (in particular, $Y$ itself is a discrete random variable, because $Y$ is always an integer). For example, if $X = 7.2$, then $Y = 7$. If $X = 12.999$, then $Y = 12$. If $X = 5.01$, then $Y = 5$. Etc., etc.

So the mass of $Y$ is exactly

$$p_Y(y) = P(Y = y) = P(y \leq X < y + 1).$$

Find an expression for the mass of $Y$. (Your expression will have $\lambda$ in it; i.e., just integrate to find the value of $P(y \leq X < y + 1)$, and then simplify.)

The mass of $Y$ is

$$P(Y = y) = \int_y^{y+1} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\bigg|_y^{y+1} = e^{-\lambda y} - e^{-\lambda(y+1)} = e^{-\lambda(1-e^{-\lambda})} = (e^{-\lambda})y(1-e^{-\lambda})$$

So if we write $q = e^{-\lambda}$ and $p = 1 - e^{-\lambda}$, then $P(Y = y) = q^y p$.

Do you recognize the mass of $Y$? (Yes, you should!) What type of random variable is $Y$? What are the parameters of $Y$? (Hint: $Y$ is one of the types of discrete random variables that we studied during the discrete random variables part of the course.)

Indeed, $Y$ is exactly the number of failures in independent Bernoulli trials until (but not including) the first success, when failure occurs with probability $q = e^{-\lambda}$ and success occurs with probability $p = 1 - e^{-\lambda}$. Thus, $Y + 1$ is exactly the number of trials until the first success (including the first success itself; that success is the “$+1$” added on). So $Y + 1$ is geometric with probability $q = e^{-\lambda}$ of failure and $p = 1 - e^{-\lambda}$ of success on each trial.

As a side note, just for those who are curious: if we define $Z = \lceil X \rceil$, by rounding $X$ up (instead of down) to the nearest integer, then

$$P(Z = z) = \int_{z-1}^{z} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\bigg|_{x=z-1}^{z} = e^{-\lambda(z-1)} - e^{-\lambda z} = (e^{-\lambda})z^{-1}(1-e^{-\lambda}) = q^{z-1} p$$

where again $q = e^{-\lambda}$ and $p = 1 - e^{-\lambda}$. So with this definition, $Z$ itself is exactly a geometric random variable with probability $q = e^{-\lambda}$ of failure or $p = 1 - e^{-\lambda}$ of success on each trial.