Chapter 28 Answers

In every answer, we use $Z \sim \mathcal{N}(0, 1)$ as a standard normal random variable.

1. Consider a randomly-selected book with the assumptions from Example 28.3, namely, the book has expected thickness $\mu = 1.125$ (in inches) and standard deviation $\sigma = .25$. If the bookseller builds a shelf that is only 1.8 inches, what is the probability that her book will fit on it?

   Let $X$ denote the thickness of the book. The probability the book will fit on the shelf is

   $\Pr(X \leq 1.8) = \Pr\left(\frac{X - 1.125}{.25} \leq \frac{1.8 - 1.125}{.25}\right) = \Pr(Z \leq 2.7) \approx .9965$

2. On a certain one-lane highway, the distance (in feet) between consecutive cars is assumed to be normally distributed with expected value $\mu = 200$ and standard deviation $\sigma = 30$. A traffic helicopter adds the distances between car 0, car 1, car 2, \ldots, car 26, and obtains the sum $X_1 + \cdots + X_{26}$ (since 27 cars have 26 gaps in between). So $X_j \sim \mathcal{N}(\mu = 200, \sigma = 30)$, and the $X_j$'s are assumed to be independent.

   Find the probability that the distance between car 0 and car 26 is less than 1 mile, i.e., find the probability that $X_1 + \cdots + X_{26} \leq 5280$.

   [There are 5280 feet in 1 mile.]

   We note that

   $\Pr(X_1 + \cdots + X_{26} \leq 5280) = \Pr(\frac{X_1 + \cdots + X_{26} - 5200}{\sqrt{23400}} \leq \frac{5280 - 5200}{\sqrt{23400}})$

   $\approx \Pr(Z \leq .52)$

   $\approx .6985$
3. As in Question 3 on Problem Set 27, assume that the annual precipitation in West Lafayette, Indiana, is normally distributed, with expected value \( \mu = 36.3 \) inches and variance \( \sigma^2 = 8.41 \). Also assume that the amount of precipitation is independent in distinct years.

Let \( X_1, \ldots, X_{10} \) denote the precipitation in the 10 distinct years of a decade. Find the probability that the total rainfall during the decade exceeds 380 inches.

We first compute the expected value:

\[
E[X_1 + \cdots + X_{10}] = E[X_1] + \cdots + E[X_{10}] = 36.3 + \cdots + 36.3 = (10)(36.3) = 363
\]

and, since the \( X_j \)'s are independent, we can also simply add the variances as follows:

\[
\text{Var}(X_1 + \cdots + X_{10}) = \text{Var}(X_1) + \cdots + \text{Var}(X_{10}) = 8.41 + \cdots + 8.41 = (10)(8.41) = 84.1
\]

So the probability that the total rainfall in the decade exceeds 380 inches is

\[
P(X_1 + \cdots + X_{10} \geq 380) = P\left(\frac{X_1 + \cdots + X_{10} - 363}{\sqrt{84.1}} \geq \frac{380 - 363}{\sqrt{84.1}}\right)
\]

\[
= P\left(Z \geq \frac{17}{\sqrt{84.1}}\right)
\]

\[
\approx P(Z \geq 1.85)
\]

\[
= 1 - P(Z \leq 1.85)
\]

\[
\approx 1 - .9678
\]

\[
= .0322
\]
4. As in Question 4 on Problem Set 27, assume that the height (in inches) of an American female is normal with expected value $\mu_1 = 64$ and standard deviation $\sigma_1 = 12.8$. Also assume that the height of an American male is normal with expected value $\mu_2 = 70$ and standard deviation $\sigma_2 = 14.0$. Let $X$ denote the female’s height and $Y$ denote the male’s height.

[Notice: Since $X$ is normal, then $aX + b$ is normal for all $a, b$; in particular, with $a = -1$ and $b = 0$, we know $-X$ is normal too. Think: What is $E[-X]$? What is Var($-X$)? Also, the sum of two normal random variables is normal, so $Y + (-X)$, a.k.a., $Y - X$, is normal too.]

Now, very carefully, find the expected value, variance, and standard deviation of $Y - X$.

The expected value of $Y - X$ is


Since $X$ and $Y$ are independent (and thus $Y$ and $-X$ are too), the variance of $Y - X$ is:

$$\text{Var}(Y - X) = \text{Var}(Y + (-X)) = \text{Var}(Y) + \text{Var}(-X) = \text{Var}(Y) + \text{Var}(X) = 14^2 + 12.8^2 = 359.84.$$

(We used the fact that $\text{Var}(-X) = (-1)^2\text{Var}(X) = \text{Var}(X)$.)

The standard deviation of $Y - X$ is $\sqrt{\text{Var}(Y - X)} = \sqrt{359.84} \approx 18.97$.

Find the probability that a randomly-selected male is taller than a randomly-selected female.

The probability that a randomly-selected male is taller than a randomly-selected female is

$$P(Y > X) = P(Y - X > 0)$$
$$= P\left(\frac{Y - X - 6}{18.97} > \frac{0 - 6}{18.97}\right)$$
$$\approx P(Z > -0.32)$$
$$= P(Z < 0.32)$$
$$\approx 0.6255$$
5. Review question: [This one is not about normal random variables.] David works at a customer call center. He talks to customers on the telephone. The length (in hours) of each conversation has density \( f_X(x) = 3e^{-3x} \) for \( x > 0 \) and \( f_X(x) = 0 \) otherwise. The lengths of calls are independent. As soon as one conversation is finished, he hangs up the phone, and immediately picks up the phone again to start another call (i.e., there are no gaps in between the calls). Thus, if he conducts \( n \) phone calls in a row, the total amount of time he spends on the telephone is \( X_1 + \cdots + X_n \), where the \( X_j \)'s are independent, and each \( X_j \) has the density above.

Find the expected value, the variance, and the standard deviation, of the amount of time that he spends on the telephone during \( n \) calls, namely, \( X_1 + \cdots + X_n \).

The expected value of \( X_j \) is just \( 1/3 \), as we have computed several times in the past:

\[
E[X_j] = \int_0^\infty (x)(3)e^{-3x} \, dx = \left(-xe^{-3x} - (1/3)e^{-3x}\right)|_{x=0}^\infty = 1/3
\]

The expected value of \( X_j^2 \) is

\[
E[X_j^2] = \int_0^\infty (x^2)(3)e^{-3x} \, dx = \left(-x^2e^{-3x} - 2x(1/3)e^{-3x} - (2)(1/9)e^{-3x}\right)|_{x=0}^\infty = 2/9
\]

So the variance of \( X_j \) is

\[
\text{Var}(X_j) = E[X_j^2] - (E[X_j])^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = 1/9.
\]

Now we compute

\[
E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] = 1/3 + \cdots + 1/3 = n/3.
\]

Since the \( X_j \)'s are independent, we can add the variances too:

\[
\text{Var}(X_1 + \cdots + X_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n) = 1/9 + \cdots + 1/9 = n/9.
\]

Finally, the standard deviation of \( X_1 + \cdots + X_n \) is \( \sqrt{\text{Var}(X_1 + \cdots + X_n)} = \sqrt{n/9} \).

What is the probability that his completes the first two calls within the first 1 hour? In other words, what is \( P(X_1 + X_2 \leq 1) \)?

Since \( X_1 \) and \( X_2 \) are independent, then their joint density if \( f_{X_1,X_2}(x_1, x_2) = 3e^{-3x_1}3e^{-3x_2} \) for \( x_1, x_2 > 0 \), and \( f_{X_1,X_2}(x_1, x_2) = 0 \) otherwise. So

\[
P(X_1 + X_2 \leq 1) = \int_0^1 \int_0^{1-x_1} 3e^{-3x_1}3e^{-3x_2} \, dx_2 \, dx_1
\]

\[
= \int_0^1 (3e^{-3x_1})(-e^{-3x_2})|_{x_2=0}^{1-x_1} \, dx_1
\]

\[
= \int_0^1 (-3e^{-3} + 3e^{-3x_1}) \, dx_1
\]

\[
= (-3e^{-3}x_1 - e^{-3x_1})|_{x_1=0}^1
\]

\[
= 1 - 4e^{-3}
\]

\[
\approx .8009
\]