Chapter 27 Answers

In every answer, we use $Z \sim N(0,1)$ as a standard normal random variable.

1. The quantity of sugar $X$ (measured in grams) in a randomly-selected piece of candy is normally distributed, with expected value $E[X] = \mu = 22$ and variance $\text{Var}(X) = \sigma^2 = 8$.

Find the probability that a randomly-selected piece of candy has less than 20 grams of sugar.

We have

$$P(X < 20) = P \left( \frac{X - 22}{\sqrt{8}} < \frac{20 - 22}{\sqrt{8}} \right)$$

$$\approx P(Z < -0.71)$$

$$= P(Z > 0.71)$$

$$= 1 - P(Z \leq 0.71)$$

$$\approx 1 - 0.7611$$

$$= 0.2389$$

2. Consider a normal random variable $X$ with expected value $\mu$ and standard deviation $\sigma$. Find the probability that $X$ is within one standard deviation of its expected value:

$$P(\mu - \sigma \leq X \leq \mu + \sigma) = P \left( \frac{\mu - \sigma - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{\mu + \sigma - \mu}{\sigma} \right)$$

$$= P(-1 \leq Z \leq 1)$$

$$= P(Z \leq 1) - P(Z \leq -1)$$

$$= P(Z \leq 1) - P(Z \geq 1)$$

$$= P(Z \leq 1) - (1 - P(Z \leq 1))$$

$$\approx 0.8413 - (1 - 0.8413)$$

$$= 0.6826$$

Find the probability that $X$ is within two standard deviations of its expected value:

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = P \left( \frac{\mu - 2\sigma - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{\mu + 2\sigma - \mu}{\sigma} \right)$$

$$= P(-2 \leq Z \leq 2)$$

$$= P(Z \leq 2) - P(Z \leq -2)$$

$$= P(Z \leq 2) - P(Z \geq 2)$$

$$= P(Z \leq 2) - (1 - P(Z \leq 2))$$

$$\approx 0.9772 - (1 - 0.9772)$$

$$= 0.9544$$
Find the probability that $X$ is within three standard deviations of its expected value:

$$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = P \left( \frac{\mu - 3\sigma - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{\mu + 3\sigma - \mu}{\sigma} \right)$$

$$= P(-3 \leq Z \leq 3)$$

$$= P(Z \leq 3) - P(Z \leq -3)$$

$$= P(Z \leq 3) - P(Z \geq 3)$$

$$= P(Z \leq 3) - (1 - P(Z \leq 3))$$

$$\approx .9987 - (1 - .9987)$$

$$= .9974$$

3. Assume that the annual precipitation in West Lafayette, Indiana, is normally distributed, with expected value $\mu = 36.3$ inches and variance $\sigma^2 = 8.41$. A rare species of frog lives in West Lafayette. This rare species of frog is known to reproduce during the year only if the annual precipitation is between 35 and 39 inches. What is the probability that the species of frog is able to reproduce this year?

Let $X$ denote the annual precipitation this year. Then

$$P(35 \leq X \leq 39) = P \left( \frac{35 - 36.3}{\sqrt{8.41}} \leq \frac{X - 36.3}{\sqrt{8.41}} \leq \frac{39 - 36.3}{\sqrt{8.41}} \right)$$

$$\approx P(-.45 \leq X \leq .93)$$

$$= P(X \leq .93) - P(X \leq -.45)$$

$$= P(X \leq .93) - P(X \geq .45)$$

$$= P(X \leq .93) - (1 - P(X \leq .45))$$

$$\approx .8238 - (1 - .6736)$$

$$= .4974$$

4. Assume that the height of an American female is normal with expected value $\mu = 64$ and standard deviation $\sigma = 12.8$.

What is the probability that an American female’s height is 66 inches or taller?

Let $X$ denote the height of an American female. Then

$$P(X \geq 66) = P \left( \frac{X - 64}{12.8} \geq \frac{66 - 64}{12.8} \right)$$

$$\approx P(Z \geq .16)$$

$$= 1 - P(Z \leq .16)$$

$$\approx 1 - .5636$$

$$= .4364$$
The heights of 10 American females are measured (in inches). Let \( Y \) be the number of the 10 females whose height is 66 inches or taller. Find \( P(Y = 7) \).

We note that \( Y \) is Binomial with \( n = 10 \) and \( p = 0.4364 \). So

\[
P(Y = 7) = \binom{10}{7} p^7 (1 - p)^3 = \binom{10}{7} (0.4364)^7 (1 - 0.4364)^3 = 0.06476.
\]

5. Review question: If \( X \) is a continuous uniform random variable on the interval \( [a, b] \), find the \( k \)th moment of \( X \), i.e., find \( E[X^k] \).

We can directly compute

\[
E[X^k] = \int_a^b x^k \frac{1}{b - a} dx = \left. \frac{x^{k+1}}{k+1} \right|_a^b = \frac{b^{k+1} - a^{k+1}}{(k+1)(b - a)}
\]

Find the variance of \( X \), i.e., \( \text{Var}(X) \).

The expected value of \( X \) is

\[
E[X] = E[X^1] = \frac{b^{1+1} - a^{1+1}}{(1+1)(b - a)} = \frac{b^2 - a^2}{2(b - a)} = \frac{(b + a)(b - a)}{2(b - a)} = (a + b)/2.
\]

The second moment of \( X \) is

\[
E[X^2] = \frac{b^{2+1} - a^{2+1}}{(2+1)(b - a)} = \frac{b^3 - a^3}{3(b - a)} = \frac{(b - a)(a^2 + ab + b^2)}{3(b - a)} = (a^2 + ab + b^2)/3.
\]

The expected value of \( X \) is

\[
\text{Var}(X) = E[X^2] - (E[X])^2
\]

\[
= \frac{a^2 + ab + b^2}{3} - \left( \frac{a + b}{2} \right)^2
\]

\[
= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12}
\]

\[
= \frac{a^2 - 2ab + b^2}{12}
\]

\[
= \frac{(b - a)^2}{12}
\]