Chapter 26 Problems

1. A painter accidentally splashes three drops of paint onto a nearby picture. The locations of the drops are independent. Fortunately, if a drop lands on the glass, it can be easily removed... however, if one or more of the drops lands on the frame of the picture, the frame is ruined. The dimensions of the glass and frame are given in the figure below. What is the probability that the frame is ruined?

When a particular drop lands on the glass itself, then joint density of its location is assumed to be uniform (because we were not told otherwise). So the probability that it lands on the glass itself—not on the frame—is equal to the area of the glass, namely, \((7.5)(10) = 75\), divided by the area of the glass plus its frame, namely, \((8.5)(11) = 93.5\), i.e., the probability is \(75/93.5 \approx 0.802139\). So the probability that all of the drops fall on the glass itself, i.e., the frame is saved, is \((75/93.5)^3 \approx 0.516118\). So the probability that the frame is ruined is \(1 - (75/93.5)^3 \approx 0.483882\).

2. A photographer is taking pictures of runners in a race. The density of the distance between each runner and the photographer (at the time of the picture) is constant on the interval \([1, 6]\), measured in feet. (Assume that the locations of the runners are independent.) The photographer’s “very good” pictures are taken when the object is less than 3 feet away. What is the probability that none of the photographers’ pictures are “very good”?

The probability that a particular picture is not “very good” is equal to the length of interval for which the corresponding picture is not “very good”, i.e., the length of \([3, 6]\), which is 3, divided by the length of the entire interval for pictures, i.e., the length of \([1, 6]\), which is 5. So the probability that a particular picture is not “very good” is \(3/5\). So the probability that all of the pictures are not “very good” is \(3/5\) raised to the number of runners. I.e., if there are \(n\) runners, the probability is \((3/5)^n\).

3. For the scenario in Question 2, if \(X, Y, Z\) are the distances of the photographer from the runners Alice, Bob, and Christine, respectively, then find the expected value of the smallest of these three distances. I.e., find \(E[\min(X, Y, Z)]\).

There are six possible orderings of the values \(X, Y, Z\). So, if we let \(U, V, W\) be the smallest, middle, and largest of the values, respectively, then \(U, V, W\) has joint density

\[
f_{U,V,W}(u, v, w) = 6(1/5)^3 \quad \text{for } 1 \leq u \leq v \leq w \leq 6,
\]

and \(f_{U,V,W}(u, v, w) = 0\) otherwise. Also note that \(U\) is the smallest of the three distances.
So the expected value of $U$ is

$$E[U] = \int_1^6 \int_1^6 \int_1^6 (u)(6)(1/5)^3 \,dw\,dv\,du$$

$$= \int_1^6 \int_1^6 (u)(6)(1/5)^3w|_{w=v} \,dv\,du$$

$$= \int_1^6 \int_1^6 (u)(6)(1/5)^3(6-v) \,dv\,du$$

$$= \int_1^6 (6)(1/5)^3(6v - v^2/2)|_{v=u} \,du$$

$$= (6)(1/5)^3(18u^2/2 - 6u^3/3 + u^4/8)|_{u=1}$$

$$= 9/4$$

$$= 2.25$$

4. Kelly throws a dart at a circular dartboard of radius 2 feet. Let $X$ and $Y$ denote the location where the dart lands. Assume that $-2 \leq X \leq 2$ and $-2 \leq Y \leq 2$ and $X^2 + Y^2 \leq 4$, i.e., the dart lands on the dartboard. Moreover, assume that the dart’s location is uniform on the dartboard, i.e.,

$$f_{X,Y}(x, y) = 1/(4\pi) \quad \text{if } x, y \text{ are on the dartboard, i.e., } x^2 + y^2 \leq 4,$$

and $f_{X,Y} = 0$ otherwise.

Let $D = \sqrt{X^2 + Y^2}$ be the distance of the dart to the center of the dartboard. Find $E[D]$.

We first find the cumulative distribution function of $D$, namely,

$$F_D(a) = P(D \leq a) = \frac{\pi a^2}{\pi 2^2} = a^2/4, \quad \text{for } 0 \leq a \leq 2,$$

so the density of $D$ is

$$f_D(a) = a/2 \quad \text{for } 0 \leq a \leq 2,$$

and $f_D(a) = 0$ otherwise.

Since we need to integrate next, it might look more natural to write

$$f_D(x) = x/2 \quad \text{for } 0 \leq x \leq 2,$$

and $f_D(x) = 0$ otherwise, where of course the “$x$” has nothing to do with random variable $X$; it just looks nicer to our eyes to integrate. So we get

$$E[D] = \int_{-\infty}^{\infty} x f_D(x) \,dx = \int_0^2 (x)(x/2) \,dx = \int_0^2 x^2/2 \,dx = 2^3/6 = 8/6 = 4/3.$$