Chapter 25 Answers

1. **There’s still a fly in the house!** Consider the two-dimensional house in Figure 1 of Problem Set 23. Suppose that a fly is still found somewhere in the house, with joint density

\[ f_{X,Y}(x, y) = \frac{1}{150} \quad \text{for } x, y \text{ in the house,} \]

and \( f_{X,Y}(x, y) = 0 \) otherwise. (We no longer need to assume \( X = 2 \) !!!)

Find \( E[Y] \).

We compute

\[
E[Y] = \int_0^{10} \int_0^{20-x} y (1/150) \, dy \, dx \\
= (1/150) \int_0^{10} (20 - x)^2/2 \, dx \\
= (1/150) \int_0^{10} (200 - 20x + x^2/2) \, dx \\
= (1/150) \left[ 200x - 2000/2 + 1000/6 \right]_{x=0}^{10} \\
= (1/150)(2000 - 2000/2 + 1000/6) \\
= 70/9
\]

Find \( \text{Var}(Y) \).

We compute

\[
E[Y^2] = \int_0^{10} \int_0^{20-x} y^2 (1/150) \, dy \, dx \\
= (1/150) \int_0^{10} (20 - x)^3/3 \, dx \\
= (1/150) \int_0^{10} (8000/3 - 400x + 20x^2 - x^3/3) \, dx \\
= (1/150) \left[ 8000x/3 - 400x^2/2 + 20x^3/3 - x^4/12 \right]_{x=0}^{10} \\
= (1/150)(80000/3 - 40000/2 + 20000/3 - 10000/12) \\
= 250/3
\]

Thus,

\[
\text{Var}(Y) = E[Y^2] - (E[Y])^2 = 250/3 - (70/9)^2 = 1850/81.
\]
2. Every day, a student calls his mother and then (afterwards) calls his girlfriend. Let \( X \) be the time (in hours) until he calls his mother, and let \( Y \) be the time (in hours) until he calls his girlfriend. Since he always calls his mother first, then \( X < Y \). So let the joint density of the time be

\[
f_{X,Y}(x, y) = 10e^{-3x-2y}
\]

for \( 0 < x < y \), and \( f_{X,Y}(x, y) = 0 \) otherwise.

Let \( Z = |Y - X| \) be time in between the beginning of the two calls.

[Hint: \( X < Y \) always, so \( Z = |Y - X| = Y - X \) always.]

Find \( E[Z] \).

We compute the integral using “by parts” with \( u = y - x \), \( dv = 10e^{-3x-2y} \), and thus \( du = dy \) and \( v = -5e^{-3x-2y} \). We get

\[
E[Z] = \int_0^\infty \int_x^\infty (y - x)10e^{-3x-2y} \, dy \, dx
\]

\[
= \int_0^\infty \left( (y - x)(-5e^{-3x-2y})\bigg|_{y=x}^\infty - \int_x^\infty -5e^{-3x-2y} \, dy \right) \, dx
\]

\[
= \int_0^\infty - (5/2)e^{-3x-2y}\bigg|_{y=x}^\infty \, dx
\]

\[
= \int_0^\infty (5/2)e^{-5x} \, dx
\]

\[
= - (1/2)e^{-5x}\bigg|_{x=0}^\infty
\]

\[
= 1/2
\]

Find \( \text{Var}(Z) \).

We compute the inner integral here using “by parts” twice in a row, which is also called the “tabular method” that Spence demonstrated in class. We get

\[
E[Z^2] = \int_0^\infty \int_x^\infty (y - x)^210e^{-3x-2y} \, dy \, dx
\]

\[
= \int_0^\infty \left( (y - x)^2(-5e^{-3x-2y})\bigg|_{y=x}^\infty - 2(y - x)((5/2)e^{-3x-2y})\bigg|_{y=x}^\infty + 2((-5/4)e^{-3x-2y})\bigg|_{y=x}^\infty \right) \, dx
\]

\[
= \int_0^\infty 2(-5/4)e^{-5x} \, dx
\]

\[
= (1/2)e^{-5x}\bigg|_{x=0}^\infty
\]

\[
= 1/2
\]

Thus

\[
\text{Var}(Z) = E[Z^2] - (E[Z])^2 = 1/2 - (1/2)^2 = 1/4.
\]
3. Bob does not like to be low on gas, so he randomly stops to fill up his tank. He has a 14-gallon tank, and the current price of gas is $2.75 per gallon. Whenever he stops to buy gas, he also buy a candy bar for $1.30. If \( X \) is the amount of gas (in gallons) in his tank when he stops for a purchase, then \( f_X(x) = 1/14 \) for \( 0 \leq x \leq 14 \), and \( f_X(x) = 0 \) otherwise. He always fills the tank, so he will always buy \( 14 - X \) gallons.

Find the expected amount of money Bob spends on a purchase of gas and a candy bar.

Method #1: We see that \( E[X] = \int_0^{14} 1/14 \, dx = 7 \), so

\[
\]

Method #2: We can integrate:

\[
E[(2.75)(14 - X) + 1.30] = \int_0^{14} ((2.75)(14 - x) + 1.30)(1/14) \, dx = 20.55.
\]

4. Review question: Preparing for Halloween, you go to the supermarket, which has an infinitely large supply of candy (i.e., each purchase does not affect the ratio of candy that remains). You notice that 30% of the candy contains peanut butter. You dearly love peanut butter. You watch with jealousy as three children independently each select a piece of candy.

Control your hungry stomach! You hurry home to study HONR 399. You ask yourself: If \( X \) is the number of children who get candy with peanut butter, what is the mass of \( X \)?

Since the 3 children select candy independently, and their choices do not affect the ratio of candy that remains, then \( X \) is binomial with \( n = 3 \) and \( p = .30 \), so the mass of \( X \) is

\[
p_X(j) = \binom{3}{j} (.30)^j (.70)^{3-j} \quad \text{for} \; j = 0, 1, 2, 3,
\]

and \( p_X(j) = 0 \) otherwise.

You dearly love your new girlfriend too, because she just came home with candy. Exactly 6 of the 10 pieces she brought you contain peanut butter. In a hungry moment of desperation, you blindly select and eat 3 of those 10 pieces. Let \( Y \) be the number which are peanut butter (since you chose blindly, all possible selections are equally likely). What is the mass of \( Y \)?

We select \( n = 3 \) out of \( N = 10 \) objects, of which the \( M = 6 \) peanut butters are “desirable” and the other \( N - M = 4 \) non-peanut-butters are “not desirable”. So \( Y \) is hypergeometric, and thus

\[
p_Y(j) = \binom{6}{j} \binom{4}{3-j} \binom{10}{3} = \frac{\binom{M}{j} \binom{N-M}{n-j}}{\binom{N}{n}}.
\]