Chapter 23 Answers

1. **There’s a fly in the house!** Consider the two-dimensional house in Figure 1.

   Suppose that a fly is found somewhere in the house, with joint density
   \[ f_{X,Y}(x,y) = \frac{1}{150} \text{ for } x,y \text{ in the house}, \]

   and \( f_{X,Y}(x,y) = 0 \) otherwise.

   Find the conditional density of \( Y \) given that \( X = 2 \).

   **Method #1.** Given that \( X = 2 \), then \( 0 \leq Y \leq 18 \). Since the joint density is constant, then the conditional density of \( Y \) on the line segment \([0,18]\) must be constant too. So \( f_{Y|X}(y \mid 2) = 1/18 \) for \( 0 \leq y \leq 18 \) and \( f_{Y|X}(y \mid 2) = 0 \) otherwise.

   **Method #2.** For \( 0 \leq x \leq 10 \), the density \( f_X(x) \) of \( X \) is
   \[ f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \int_0^{20-x} 1/150 \, dy = (20-x)/150 \]

   Thus, \( f_X(2) = (20-2)/150 = 18/150 \). So, for \( 0 \leq y \leq 18 \), we have
   \[ f_{Y|X}(y \mid 2) = f_{X,Y}(x,2)/f_X(2) = (1/150)/(18/150) = 1/18. \]

   Find the conditional probability that the fly is upstairs, i.e., \( Y > 10 \), given that \( X = 2 \), i.e., find \( P(Y > 10 \mid X = 2) \).

   **Method #1.** Since \( f_{Y|X}(y \mid 2) \) is constant for \( y \) on the interval \([0,18]\), then the conditional probability that \( Y > 10 \) (given \( X = 2 \), of course) is the length of \([10,18]\), namely 8, over the length of the whole line, namely, 18. So the desired conditional probability is \( 8/18 \).

   **Method #2.** So \( f_{Y|X}(y \mid 2) = 1/18 \) for \( 0 \leq y \leq 18 \) and \( f_{Y|X}(y \mid 2) = 0 \) otherwise, then
   \[ P(Y > 10 \mid X = 2) = \int_{10}^{\infty} f_{Y|X}(y \mid 2) \, dy = \int_{10}^{18} 1/18 \, dy = 8/18. \]
2. Every day, a student calls his mother and then (afterwards) calls his girlfriend. Let $X$ be the time (in hours) until he calls his mother, and let $Y$ be the time (in hours) until he calls his girlfriend. Since he always calls his mother first, then $X < Y$. So let the joint density of the time be

$$f_{X,Y}(x, y) = 10e^{-3x-2y} \quad \text{for } 0 < x < y,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

Given that $X = 1/2$, find the conditional probability that $Y > 2/3$. In other words, find $P(Y > 2/3 \mid X = 1/2)$.

First we find the density of $X$ by integrating the $y$’s away. For $x > 0$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy$$

$$= \int_{x}^{\infty} 10e^{-3x-2y} \, dy$$

$$= -5e^{-3x-2y} \bigg|_{y=x}^{\infty}$$

$$= 5e^{-3x-2x}$$

$$= 5e^{-5x}$$

So, in particular, $f_X(1/2) = 5e^{-5/2}$.

Now the conditional density of $Y$, given $X = 1/2$, is

$$f_{Y \mid X}(y \mid 1/2) = \frac{f_{X,Y}(y, 1/2)}{f_X(1/2)} = \frac{10e^{-3(1/2)-2y}}{5e^{-5/2}} = 2e^{1-2y} \quad \text{for } y > 1/2,$$

and $f_{Y \mid X}(y \mid 1/2) = 0$ otherwise.

Finally,

$$P(Y > 2/3 \mid X = 2) = \int_{2/3}^{\infty} f_{Y \mid X}(y \mid 1/2) \, dy = \int_{2/3}^{\infty} 2e^{1-2y} \, dy$$

$$= -e^{1-2y} \bigg|_{y=2/3}^{\infty}$$

$$= e^{1-2(2/3)}$$

$$= e^{-1/3}$$

$$\approx 0.7165313.$$
3. The State of Wyoming is approximately shaped like a rectangle. For the purposes of this problem, assume that Wyoming is exactly a rectangle, and that a person’s location in Wyoming is $0 \leq X \leq 350$ and $0 \leq Y \leq 276$. Highway 80 is nearly straight, so for the purposes of this problem, assume that Highway 80 runs perfectly along the east-to-west line $Y = 30$. Assume that the joint density of a person’s location is

$$f_{X,Y}(x, y) = \frac{1}{(276)(350)}$$

for $0 \leq x \leq 350$ and $0 \leq y \leq 276$, and $f_{X,Y}(x, y) = 0$ otherwise.

Given that the person is located on Highway 80, what is the probability that he is within 10 miles of a State border? (He could be either west or east; please take both into account.)

**Method #1.** Since the joint density is constant, then the density $f_{X|Y}(x \mid 30)$ of $X$ given $Y = 30$ must be constant too, on the interval $[0, 350]$, and we want the probability that $X$ is within the first 10 units—which is $10/350$, again, since the density is constant—or the probability that $X$ is within the last 10 units—which is again $10/350$, again, since the density is constant. These are disjoint possibilities, so the answer is $10/350 + 10/350 = 2/35$.

**Method #2.** We compute the density of $Y$ by integrating the $x$’s away. For $0 \leq Y \leq 276$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx = \int_{0}^{350} \frac{1}{(276)(350)} = 1/276.$$

In particular, $f_Y(30) = 1/276$.

Thus, for $0 \leq x \leq 350$, we have $f_{X|Y}(x \mid 30) = \frac{f_{X,Y}(x, 30)}{f_Y(30)} = \frac{1/((276)(350))}{1/276} = 1/350$. Finally,

$$P(0 \leq X \leq 10 \text{ or } 340 \leq X \leq 350) = \int_{0}^{10} f_{X|Y}(x \mid 30) \, dx + \int_{340}^{350} f_{X|Y}(x \mid 30) \, dx$$

$$= \int_{0}^{10} 1/350 \, dx + \int_{340}^{350} 1/350 \, dx$$

$$= 10/350 + 10/350$$

$$= 2/35.$$
4. Consider the semicircle with radius 2 seen in Figure 3. A dancer is randomly located in the semicircle, with the joint density of their location to be

\[ f_{X,Y}(x, y) = \frac{1}{2\pi} \text{ if } x, y \text{ is in the semicircle,} \]

and \( f_{X,Y}(x, y) = 0 \) otherwise.

Find the probability that the person is in the region where \(-1 \leq X \leq 1\) and \(0 \leq Y \leq 1\).

Method #1. The entire area \(-1 \leq X \leq 1\) and \(0 \leq Y \leq 1\) is enclosed in the semicircle. The joint density is constant. So the desired probability is the area of the rectangle, \((2)(1) = 2\) divided by the entire area, \(2\pi\). So the desired probability is \(\frac{2}{2\pi} = \frac{1}{\pi} \approx 0.31831\).

Method #2. We can just integrate over the whole rectangle, since it is entirely within the bounds of the semicircle:

\[
\int_{-1}^{1} \int_{0}^{1} \frac{1}{2\pi} \, dy \, dx = \int_{-1}^{1} \frac{1}{2\pi} \, dx = \frac{2}{2\pi} = 1/\pi \approx 0.31831
\]

Figure 3: Picture of a semicircle-shaped stage