Chapter 22 Answers

1. When an emergency occurs, the response time (in hours) of the first police car is a random variable $X$ with density

$$f_X(x) = 12e^{-12x} \quad \text{for } x > 0,$$

and $f_X(x) = 0$ otherwise. The response time (in hours) of the first fire engine is a random variable $Y$ with density

$$f_Y(y) = 10e^{-10y} \quad \text{for } y > 0,$$

and $f_Y(y) = 0$ otherwise.

Find the probability that the first police car arrives before the first fire engine.

Since the arrival times can be assumed to be independent (i.e., no dependence is mentioned between $X$ and $Y$), then the joint density is $f_{X,Y}(x, y) = 12e^{-12x}10e^{-10y} = 120e^{-12x-10y}$, for $0 < x$ and $0 < y$, and $f_{X,Y}(X, Y) = 0$ otherwise. The region of integration of the joint density, to get the desired probability, is shown in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{region}
\caption{Region the shows when the first police car arrives before the first fire engine}
\end{figure}

\textit{Method #1.} We can integrate over $x$ as the outer integral and over $y$ as the inner integral:

$$P(X < Y) = \int_0^\infty \int_x^\infty 120e^{-12x-10y} \, dy \, dx$$

$$= \int_0^\infty \left[ -12e^{-12x-10y} \right]_x^\infty \, dx$$

$$= \int_0^\infty 12e^{-22x} \, dx$$

$$= \left[ -\frac{12}{22}e^{-22x} \right]_0^\infty$$

$$= \frac{12}{22}$$

$$= 6/11$$
Method #2. We can integrate over $y$ as the outer integral and over $x$ as the inner integral:

\[
P(X < Y) = \int_0^\infty \int_0^y 120e^{-12x-10y} \, dx \, dy
\]

\[
= \int_0^\infty -10e^{-12x-10y} \bigg|_0^y \, dy
\]

\[
= \int_0^\infty -10(e^{-22y} - e^{-10y}) \, dy
\]

\[
= -10 \left( \frac{1}{-22} e^{-22y} - \frac{1}{-10} e^{-10y} \right) \bigg|_y^0
\]

\[
= -10 \left( \frac{1}{22} - \frac{1}{10} \right)
\]

\[
= 6/11
\]

2. Consider the cumulative distribution function

\[
F_X(x) = \begin{cases} 
\frac{1}{4}x + \frac{1}{2} & \text{for } -2 < x < 2, \\
0 & \text{for } x \leq -2, \\
1 & \text{for } x \geq 2.
\end{cases}
\]

Draw a picture of $F_X(x)$. Now find the density $f_X(x)$.

![Figure 2: The cumulative distribution function $F_X(x)$](image)

The density $f_X(x)$ of a random variable is found by differentiating the cumulative distribution function, $F_X(x)$. In this case,

\[
f_X(x) = F'_X(x) = \frac{d}{dx} \left( \frac{1}{4}x + \frac{1}{2} \right) = \frac{1}{4} \quad \text{for } -2 < x < 2,
\]

and $f_X(x) = 0$ otherwise.
3. Police cars are randomly stationed throughout the town. When an emergency occurs, the distance a police car must travel north or south is a random variable \( X \) with density
\[
f_X(x) = \frac{1}{6} \quad \text{for } 0 \leq x \leq 6,
\]
and \( f_X(x) = 0 \) otherwise. The distance a police car must travel east or west is a random variable \( Y \) with density
\[
f_Y(y) = \frac{1}{6} \quad \text{for } 0 \leq y \leq 6,
\]
and \( f_Y(y) = 0 \) otherwise. Assume that \( X \) and \( Y \) are independent.

So \( X + Y \) is the total distance traveled. Find the probability that \( X + Y \leq 4 \).

[Hint: First find the joint density \( f_{X,Y}(x, y) \) of \( X \) and \( Y \), and draw a picture for where the joint density is defined. If you write an integral for the probability, the integrand is constant. So you can compute the desired area in your picture, divided by the total area.]

Since \( X \) and \( Y \) are independent, the joint density of \( X \) and \( Y \) is
\[
f_{X,Y}(x, y) = f_X(x)f_Y(y) = (1/6)(1/6) = 1/36 \quad \text{for } 0 \leq x \leq 6 \text{ and } 0 \leq y \leq y,
\]
and \( f_{X,Y}(x, y) = 0 \) otherwise. So the integrand is constant. So we have at least two ways that we can compute the integral.

Method #1. The entire area where the joint density is positive is 36. The desired region is shown in Figure 3; this region has area \((1/2)(4)(4) = 8\). So the desired probability is \( 8/36 = 2/9 \). Method #2. We can integrate:

![Figure 3: Region where the total area travelled, \( X + Y \), is less than 4.](image)

\[
P(X + Y \leq 4) = \int_0^4 \int_0^{4-x} 1/36 \, dy \, dx
\]
\[
= \int_0^4 (4 - x)/36 \, dx
\]
\[
= (4x - x^2/2)/36 \big|_{x=0}^4
\]
\[
= (4)(4) - 4^2/2)/36
\]
\[
= 8/36
\]
\[
= 2/9
\]
4. Consider $X$ and $Y$ with joint density

\[ f_{X,Y}(x, y) = \frac{3}{32} x^3 y^2 \quad \text{for } 0 < x < 2 \text{ and } 0 < y < 2, \]

and $f_{X,Y}(x, y) = 0$ otherwise.

Find the density of $Y$.

We integrate $X$ out of the picture, as follows:

For $y \leq 0$ or $y \geq 2$, we see that $f_{X,Y}(x, y) = 0$ so $f_Y(y) = 0$ too.

Now consider $0 < y < 2$. We compute

\[
\begin{align*}
    f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \\
    &= \int_0^2 \frac{3}{32} x^3 y^2 \, dx \\
    &= \left( \frac{3}{32} \right) \left( \frac{x^4}{4} \right) y^2 \bigg|_{x=0} \\
    &= \left( \frac{3}{32} \right) \left( \frac{2^4}{4} \right) y^2 \\
    &= \frac{3}{8} y^2
\end{align*}
\]

So we conclude that the density of $Y$ is

\[ f_Y(y) = \frac{3}{8} y^2 \quad \text{for } 0 < y < 2, \]

and $f_Y(y) = 0$ otherwise.