Chapter 21 Answers

1. Consider the random variables $X$ and $Y$ defined in Example 21.1, namely, $X$ is Alice’s waiting time, and $Y$ is Bob’s waiting time. Let $W = \max(X, Y)$, i.e., $W$ is either Alice’s waiting time or Bob’s waiting time, whichever is larger!

Find $F_W(w) = P(W \leq w)$, the cumulative distribution function of $W$. This is equal to $P(\max(X, Y) \leq w)$, i.e., the probability that Alice waits less than $w$ seconds and Bob waits less than $w$ seconds.

Since $X > 0$ always and $Y > 0$ always, then $W > 0$ too. So, for $w \leq 0$, we have $F_W(w) = P(W \leq w) = 0$.

Now consider $w > 0$. We compute:

\[
F_W(w) = P(W \leq w) \\
= P(\max(X, Z) \leq w) \\
= P(X \leq w \text{ and } Y \leq w) \\
= \int_0^w \int_0^w 15e^{-3x-5y} \, dy \, dx \\
= \int_0^w -3e^{-3x-5y}\big|_{y=0}^w \, dx \\
= \int_0^w (-3e^{-3x-5w} + 3e^{-3x}) \, dx \\
= (e^{-3x-5w} - e^{-3x})\big|_{x=0}^w \\
= e^{-8w} - e^{-5w} - e^{-3w} + 1
\]

So, in summary, the cumulative distribution function of $W$ is

\[
F_W(w) = e^{-8w} - e^{-5w} - e^{-3w} + 1 \quad \text{for } w > 0,
\]

and $F_W(w) = 0$ otherwise.

2. A bird lands in a grassy region described as follows: $0 \leq x$, and $0 \leq y$, and $x + y \leq 10$. This region is shown in the figure below. Let $X$ and $Y$ be the coordinates of the bird’s landing. Assume that $X$ and $Y$ have the joint density

\[
f_{X,Y}(x, y) = 1/50 \quad \text{for } 0 \leq x \text{ and } 0 \leq y \text{ and } x + y \leq 10,
\]

and $f_{X,Y}(x, y) = 0$ otherwise.

Find $P(X \leq 7 \text{ and } Y \leq 7)$. 

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Method #1. Since the integrand is constant, then—as discussed in class—divide the desired area \((8 + 12 + 12 + 9 = 41)\) over the entire area \((50)\) to get the desired probability, \(\frac{41}{50}\).

Method #2. We can integrate carefully, as in Figure 1. We must use two regions of integration (see the picture), because the limits for \(y\) change, depending on the value of \(x\):

\[
P(X \leq 7 \text{ and } Y \leq 7) = \int_0^3 \int_0^7 \frac{1}{50} dy \, dx + \int_3^7 \int_0^{10-x} \frac{1}{50} dy \, dx
\]

\[
= \int_0^3 \frac{y}{50} \bigg|_{y=0}^7 \, dx + \int_3^7 \frac{y}{50} \bigg|_{y=0}^{10-x} \, dx
\]

\[
= \int_0^3 \frac{7}{50} \, dx + \int_3^7 \frac{10 - x}{50} \, dx
\]

\[
= \frac{7x}{50} \bigg|_0^3 + \frac{10x - x^2/2}{50} \bigg|_3^7 = \frac{21}{50} + \frac{(10)(7) - 7^2/2}{50} - \frac{(10)(3) - 3^2/2}{50}
\]

\[
= 41/50
\]

Method #3. We could find the probability of the complement:

\[
P(X \leq 7 \text{ and } Y \leq 7) = 1 - P(X > 7 \text{ or } Y > 7)
\]

\[
= 1 - [P(X > 7) + P(Y > 7) - P(X > 7 \text{ and } Y > 7)]
\]

In this problem, \(X > 7\) and \(Y > 7\) cannot happen simultaneously, so \(P(X > 7 \text{ and } Y > 7) = 0\). Also, by symmetry, \(P(X > 7)\) and \(P(Y > 7)\) are the same. We compute

\[
P(X > 7) = \int_7^{10} \int_0^{10-x} \frac{1}{50} dy \, dx = \int_7^{10} \frac{y}{50} \bigg|_{y=0}^{10-x} \, dx = \int_7^{10} \frac{10 - x}{50} \, dx = \frac{10x - x^2/2}{50} \bigg|_{x=7}^{10} = 4.5/50
\]

So we conclude \(P(X \leq 7 \text{ and } Y \leq 7) = 1 - (4.5/50) - (4.5/50) = 1 - 9/50 = 41/50\).
3. Consider random variables $X$ and $Y$ with joint density

$$f_{X,Y}(x, y) = \frac{1}{8}(1 - x^2)(3 - y)$$

for $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$, and $f_{X,Y}(x, y) = 0$ otherwise. Find the probability that $X$ and $Y$ are both negative.

The region of integration is given in Figure 2. So we just compute:

$$P(X < 0 \text{ and } Y < 0) = \int_{-1}^{0} \int_{-1}^{0} \frac{1}{8}(1 - x^2)(3 - y) \, dy \, dx$$

$$= \int_{-1}^{0} \frac{1}{8}(1 - x^2) \left(3y - \frac{y^3}{2}\right) \bigg|_{y=-1} dx$$

$$= \int_{-1}^{0} \frac{1}{8}(1 - x^2)(7/2) \, dx$$

$$= \frac{7}{16} \int_{-1}^{0} (1 - x^2) \, dx$$

$$= \frac{7}{16} (x - \frac{x^3}{3}) \bigg|_{x=-1}$$

$$= (7/16)(2/3)$$

$$= 7/24$$

Figure 2: The region where $X$ and $Y$ are both negative
4. Freddy and Jane have entered a game in which they each win between 0 and 2 dollars. If $X$ is the amount Freddy wins, and $Y$ is the amount that Jane wins, they believe that the joint density of their winnings will be

$$f_{X,Y}(x, y) = \frac{1}{4}xy \quad \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

Find the probability that their combined winnings exceed 2, i.e., find $P(X + Y > 2)$.

We show the region of integration in Figure 3. Then we just compute

$$P(X + Y > 2) = \int_0^2 \int_{2-x}^2 \frac{1}{4}xy \, dy \, dx$$

$$= \int_0^2 \left[ \frac{1}{4}xy^2/2 \right]_{y=2-x}^2 \, dx$$

$$= \int_0^2 \frac{1}{8}x(2^2 - (2 - x)^2) \, dx$$

$$= \int_0^2 \frac{1}{8}(4x^2 - x^3) \, dx$$

$$= \frac{1}{8} \left[ \frac{4x^3}{3} - \frac{x^4}{4} \right]_0^2$$

$$= \frac{1}{8} \left( \frac{4(2)^3}{3} - \frac{2^4}{4} \right)$$

$$= \frac{5}{6}$$

![Figure 3: The region where $X + Y > 2$](image)