Chapter 21 Problems

1. Consider the random variables $X$ and $Y$ defined in Example 21.1, namely, $X$ is Alice’s waiting time, and $Y$ is Bob’s waiting time. Let $W = \max(X, Y)$, i.e., $W$ is either Alice’s waiting time or Bob’s waiting time, whichever is larger!

Find $F_W(w) = P(W \leq w)$, the cumulative distribution function of $W$. This is equal to $P(\max(X, Y) \leq w)$, i.e., the probability that Alice waits less than $w$ seconds and Bob waits less than $w$ seconds.
2. A bird lands in a grassy region described as follows: $0 \leq x$, and $0 \leq y$, and $x + y \leq 10$. This region is shown in the figure below. Let $X$ and $Y$ be the coordinates of the bird’s landing. Assume that $X$ and $Y$ have the joint density

$$f_{X,Y}(x, y) = \frac{1}{50} \quad \text{for } 0 \leq x \text{ and } 0 \leq y \text{ and } x + y \leq 10,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

Find $P(X \leq 7 \text{ and } Y \leq 7)$. 

Figure 1: The grassy region where a bird lands
3. Consider random variables $X$ and $Y$ with joint density

$$f_{X,Y}(x, y) = \frac{1}{8} (1 - x^2)(3 - y) \quad \text{for } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1,$$

and $f_{X,Y}(x, y) = 0$ otherwise. Find the probability that $X$ and $Y$ are both negative.
4. Freddy and Jane have entered a game in which they each win between 0 and 2 dollars. If $X$ is the amount Freddy wins, and $Y$ is the amount that Jane wins, they believe that the joint density of their winnings will be

$$f_{X,Y}(x, y) = \frac{1}{4}xy \quad \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

Find the probability that their combined winnings exceed 2, i.e., find $P(X + Y > 2)$. 
5. Design your own joint density $f_{X,Y}(x,y)$. Make sure that the $f_{X,Y}(x,y)$ that you design is actually a joint density, i.e., you need to check that

1. $f_{X,Y}(x,y) \geq 0$ for all $x, y$, and
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1$. 
6. Using your joint density from Question 5, calculate a probability that is interesting to you. For instance, you could calculate

\[ P(X > 3 \text{ and } Y < 7) \]

or you could calculate

\[ P(X \leq 2 \text{ and } Y \leq 2) \]

or you could calculate

\[ F_{X,Y}(a, b) = P(X \leq a \text{ and } Y \leq b) \]

Etc., etc., anything you like is OK.